The Log-Local-Open Correspondence

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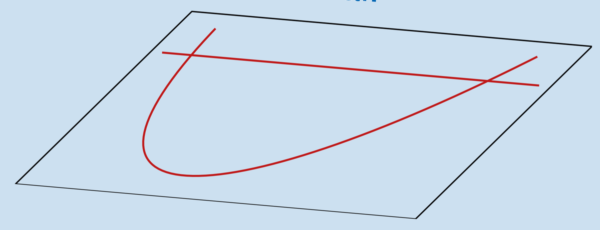
The Geometry

For a *(nef) Looijenga pair* (smooth rational projective complex surface; anticanonical singular nodal curve (with all irreducible components D_i nef))

$$(Y, D = D_1 + \ldots + D_\ell)$$

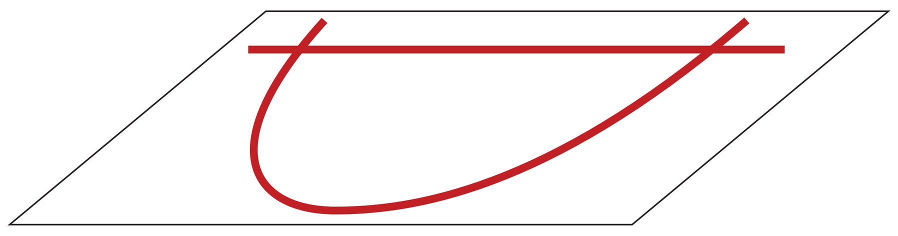
we construct...

Example: $Y = \mathbb{P}^2$ and $D = \text{line} + \text{conic}$ as below. Then $[\text{line}] + [\text{conic}] = H + 2H = |-K_{\mathbb{P}^2}|$ $\Rightarrow$ $(\mathbb{P}^2, D)$ is anti-canonical.



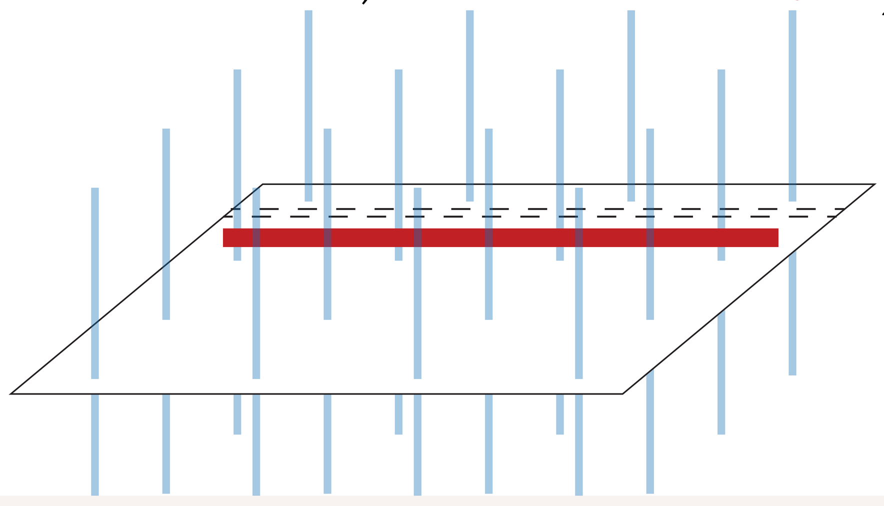
Log Calabi-Yau 2-fold

$$Y(D_1 + \ldots + D_\ell)$$



Toric Calabi-Yau 3-fold + $(\ell - 1)$ Lagrangians

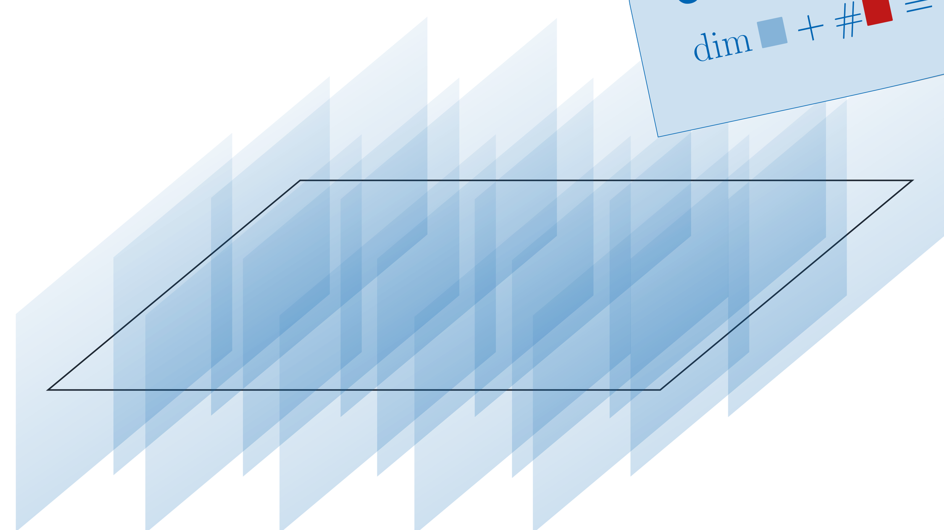
$$\left(\mathrm{Tot}\, \mathcal{O}_Y(-D_\ell)|_{Y \setminus \bigcup_{i \neq \ell} D_i}, L_1 \sqcup \ldots \sqcup L_{\ell-1}\right)$$



Local Calabi-Yau $(\ell + 2)$-fold

$$\mathrm{Tot} \bigoplus_{i=1}^{\ell} \mathcal{O}_Y(-D_i)$$

Observation: dim ■ + #■ = const

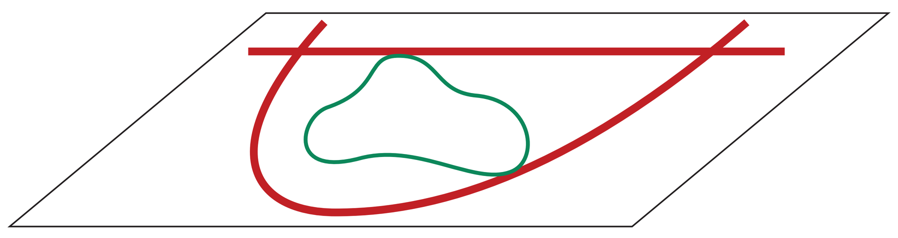


Genus zero GW Invariants

For any of the geometries X above, given $\beta \in H_2(X, \mathbb{Z})$ define the Gromov–Witten invariants

$$N^X_{0,\beta} := \text{"}\# \left\{ \begin{array}{c} \text{class } \beta, \text{ genus } g{=}0 \text{ stable maps } f{:}C \to X \\ \text{with } \cdots \text{ satisfying point conditions} \end{array} \right\}\text{"}$$

$$N^{\log}_\beta = \# \left\{ \begin{array}{c} \text{...maximum tangency} \\ (\beta \cdot D_i) \text{ at each } D_i \text{ (and a} \\ \lambda_{g=0} \text{ insertion)...} \end{array} \right\}$$

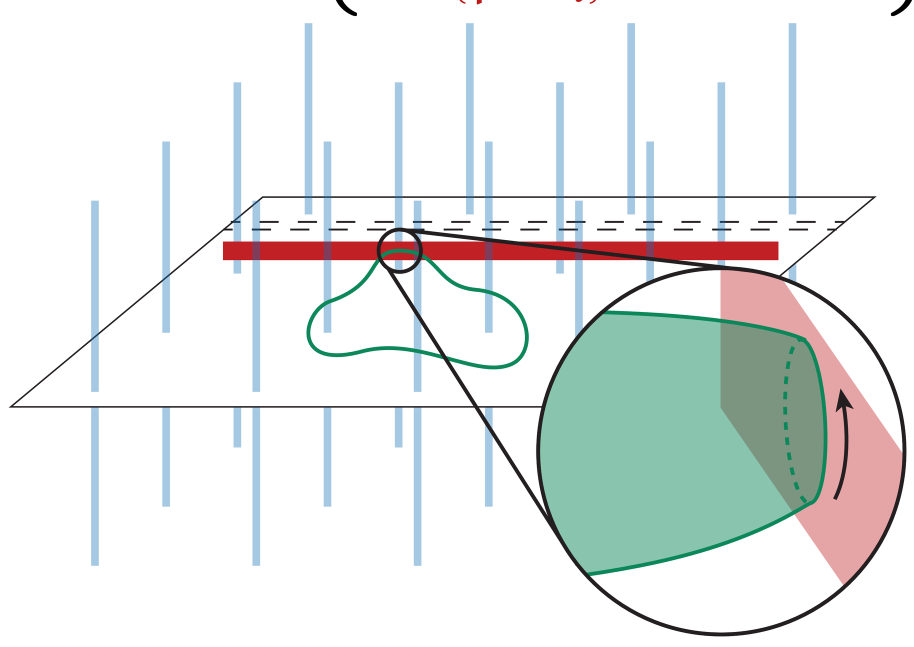


Some facts about $N^{\log}_\beta$:

↳ occur in context of GS mirror symmetry

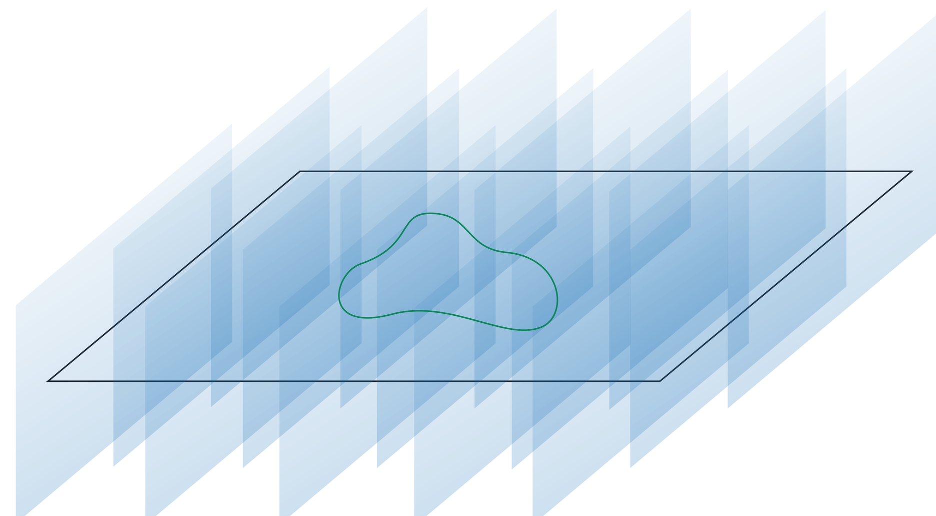
↳ computed via tropical correspondence theorems and scattering diagrams

$$N^{\text{open}}_\beta = \# \left\{ \begin{array}{c} \text{...}(\ell-1) \text{ boundaries} \\ \text{each wrapping one } L_i \\ (\beta \cdot D_i)\text{-times...} \end{array} \right\}$$



↳ computed using the topological vertex

$$N^{\text{local}}_\beta = \# \{\text{...*nothing*...}\}$$



↳ computed via Givental I, J-function mirror thms

↳ instance of twisted GW invariants which enjoy Givental reconstruction

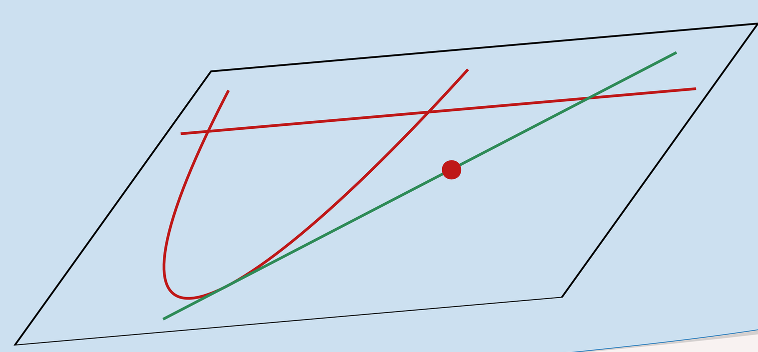
The Log-Open-Local Correspondence

$$\prod_{i=1}^{\ell} \frac{(-1)^{\beta \cdot D_i + 1}}{\beta \cdot D_i} N^{\log}_\beta = N^{\text{open}}_\beta = N^{\text{local}}_\beta$$

Proven for quasi-tame nef Looijenga pairs and conjectured for more general Looijenga pairs [1].

Example: Let's compute $N^{\log}_\beta$ for $Y(D) = \mathbb{P}^2(\text{line} + \text{conic})$ and $\beta = [\text{line}]$. Then $\beta \cdot D_1 = 1$ and $\beta \cdot D_2 = 2$ and so

$$N^{\log}_{[\text{line}]} = \# \left\{ \begin{array}{c} \text{lines in } \mathbb{P}^2 \text{ intersecting } D_1 \\ \text{and meeting } D_2 \text{ transversally} \\ \text{and passing through } \bullet \end{array} \right\} = 2$$



Higher genus

$$N^X_{g,\beta} \rightsquigarrow \mathsf{N}^X_\beta := \sum_{g \geq 0} \hbar^{2g-*} N^X_{g,\beta}$$

The all genus Log-Open Correspondence

$$\frac{(-1)^{d \cdot D_\ell + 1}}{2 \sin \frac{\hbar\, d \cdot D_\ell}{2}} \prod_{i=1}^{\ell-1} \frac{(-1)^{d \cdot D_i + 1}}{d \cdot D_i} \mathsf{N}^{\log}_\beta = \mathsf{N}^{\text{open}}_\beta$$

Proven for quasi-tame nef Looijenga pairs [1, 2] and again conjectured for more general Looijenga pairs [1].

References

[1] Pierrick Bousseau, Andrea Brini, and Michel van Garrel. Stable maps to looijenga pairs. *arXiv preprint arXiv:2011.08830*, 2020.

[2] Andrea Brini and Yannik Schüler. On quasi-tame looijenga pairs. *arXiv preprint arXiv:2201.01645*, 2022.

[3] Michel van Garrel, Tom Graber, and Helge Ruddat. Local gromov-witten invariants are log invariants. *Advances in Mathematics*, 350:860–876, 2019.