GW theory for the refined topological string

1. From physics to maths

Let X be a smooth (quasi-)proj. CY 3fold / $\mathbb{C}$.

topological string = A-twisted $\mathcal{N}=(2,2)$ σ-model coupled to 2d gravity

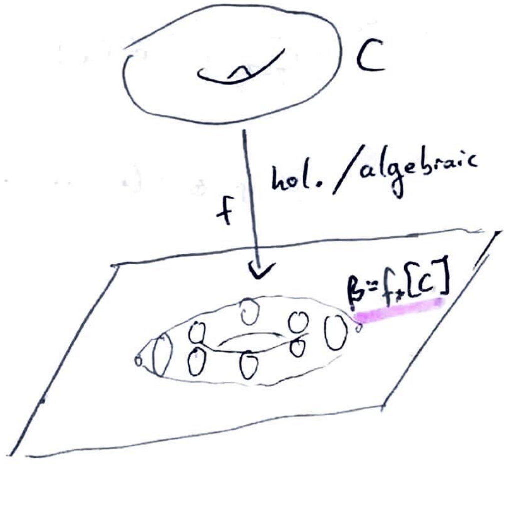
Gromov-Witten theory

$e^{-(f_*[C],\omega)}$ $\quad u^{2g_C-2}$

$$\int_{\{f\in C^\infty(C,X)\}+\text{SUSY}} \mathcal{D}f\,\mathcal{D}[C]\cdots e^{S(f,\dots,C)}\, e^{S_{EH}(C)} = \sum_{\beta\neq 0} e^{-(\beta,\omega)} \underbrace{\sum_{g\geq 0} u^{2g-2} \int_{[\overline{M}_g(X,\beta)]^{vir}} 1}_{=:GW_\beta(X)}$$

localises on $\bar{\partial} f = 0$

SUSY localisation



$$\overline{M}_g(X,\beta) = \left\{ C \xrightarrow{f} X \text{ stable map, } C \text{ a genus } g \text{ curve and } f_*[C]=\beta \right\}/\sim$$

(exp)dim $= (\dim X - 3)(1-g)$
$= 0$

2. The refinement

draw picture

M-theory on
$X \times \mathbb{C}^2 \times S^1 \circlearrowleft T \cong (\mathbb{C}^*)^2$

$\varepsilon_i :=$ T-weight on i-th factor of $\mathbb{C}^2$
$\in H_T^*(pt)$

$\downarrow$

refined topological string
$GW_\beta^{ref}(X) = ???$ $\in \mathbb{Q}[\varepsilon_1, \varepsilon_2]$

? refinement

$\circledast \quad \Big\downarrow \; \varepsilon_1 = -\varepsilon_2 = iu$

topological string
$GW_\beta(X)$ $\in \mathbb{Q}[u]$

DT-theory
SUSY gauge th.
Chern Simons,
...

Proposal: $GW_\beta^{ref}(X) = \int_{[\overline{M}(X \times \mathbb{C}^2, \beta)]^{vir}} 1$

Slogan: Count points with symmetry.

$$= \sum_{g \geq 0} \underbrace{\boxed{{}^{T}\!\!\int_{[\overline{M}_g(X \times \mathbb{C}^2, \beta)]^{vir}} 1}}_{\in H^T_{2-2g}(pt) \cong \mathbb{Q}[\varepsilon_1, \varepsilon_2]_{2g-2}}$$

Issues:

- infinite sum ← filtered by grading
- $\overline{M}_g(X \times \mathbb{C}^2, \beta) = \overline{M}_g(X, \beta) \times \mathbb{C}^2$ non-compact ← localisation
- expdim $= 2-2g < 0$. ✓

Assumption: $T \cong (\mathbb{C}^\times)^2$ satisfies

i) $K_{X\times\mathbb{C}^2}$ is fixed

ii) K_X is <u>not</u> fixed

iii) $\exists\, \mathbb{C}^\times \hookrightarrow T$ acting via

$$\begin{aligned} \mathbb{C}^\times \times \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (t, (x_1, x_2)) &\longmapsto (t x_1, t^{-1} x_2) \end{aligned}$$

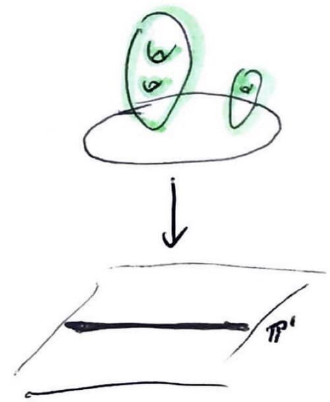
Rem: (ii) $\Rightarrow X$ non-compact

Lem: $\circledast$ holds. [Mumford '83]

Conj: $GW_\beta^{\mathrm{ref}}(X)$ does not depend on choice of T-action (if (iv) $\overline{M}_g(X,\beta)$ compact $\forall g$).

Rem: related to the choice of preferred direction for the refined top. vertex.

3. BPS integrality



Example: $X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

$\boxed{\text{degree } d \text{ covers contracted components}}$

$$GW^{\mathrm{ref}}_{d[\mathbb{P}^1]}(X) \overset{d=1}{=} \frac{1}{\varepsilon_1\varepsilon_2}\Big(\underbrace{1}_{\text{genus } 0} \underbrace{- \tfrac{1}{24}\varepsilon_1^2 - \tfrac{1}{24}\varepsilon_2^2}_{\text{genus } 1} \underbrace{+ \tfrac{7}{5760}\varepsilon_1^4 + \tfrac{1}{576}\varepsilon_1^2\varepsilon_2^2 + \tfrac{7}{5760}\varepsilon_2^4 + \dots}_{\text{genus } 2}\Big)$$

$$= \frac{1}{d}\,\frac{1}{2\sinh\frac{d\varepsilon_1}{2}\cdot 2\sinh\frac{d\varepsilon_2}{2}}$$

=

Write expansion first.

for fairly general T-action

Conj: ① $GW_\beta(X)$ lifts to a rational function in $e^{\varepsilon_1}, e^{\varepsilon_2}$

② ...with denom. as in example & num. w/ integer coefficients.

↑ for general X. if (i)–(iv) hold.

Thm I: [Brini–S] ① holds for $X = K_S$ a local del Pezzo in $\varepsilon_2 = 0$ limit.

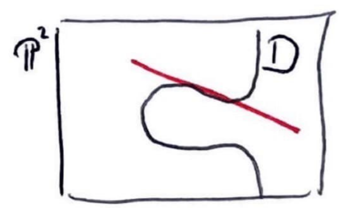
①, ② hold for $K_{\mathbb{P}^2}$ in $\varepsilon_2 = 0$ limit.

Rem: The limit $\varepsilon_1 = -\varepsilon_2 = iu$ is proven by [Ionel–Parker, Doan–Ionel–Walpuski]

Rem: For $K_{\mathbb{P}^2}$: rational lift = Maulik–Toda type BPS invariant

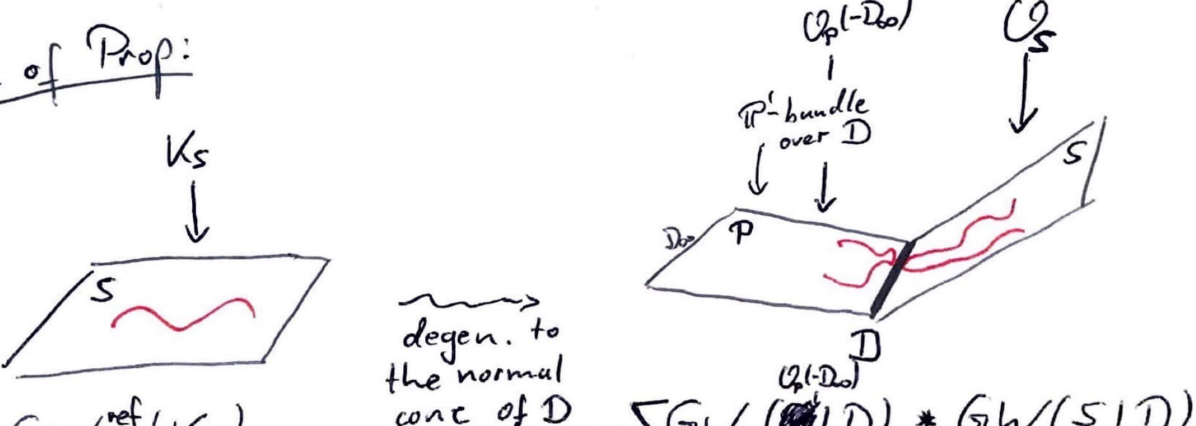
Prop: For S a del Pezzo surface & D a smooth anticanonical curve we have

$$\varepsilon_2 \, GW^{ref}_\beta(K_S) = \frac{(-1)^{D\cdot\beta+1}}{D\cdot\beta} \cdot \sum_{g\geq 0} \varepsilon_1^{2g-1} \cdot \#\left\{\begin{matrix}\text{genus } g, \text{ class } \beta \\ \text{maximal tangents} \\ \text{to } D\end{matrix}\right\}$$



Thm: [Boussau] Thm I holds for rhs.

Pf of Prop:



$$GW^{ref}_\beta(K_S) = \sum GW_*(\mathcal{O}_P(-D_\infty) | D) * GW(S|D)$$ □