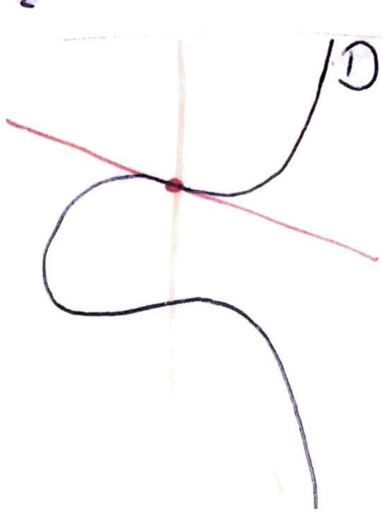
Tangents to the nodal cubic

Degree one

Fact ②: $V(H)$ & $C = V(f)$ intersect transversely

$D: F = y^2 z - x^3 - x^2 z - 1 z^3 = 0$

$\cancel{\mathbb{C}^2} \to \mathbb{P}^2$



$H := \det \operatorname{Hess}(f)_p = 0$

$\Updownarrow$ Fact ①

$n_1 = \#\{p \in D : (T_p \cdot C)_p = 3\}$

$= \sum_{p \in D \cap V(H)} \underbrace{(D \cdot V(H))_p}_{=1}$

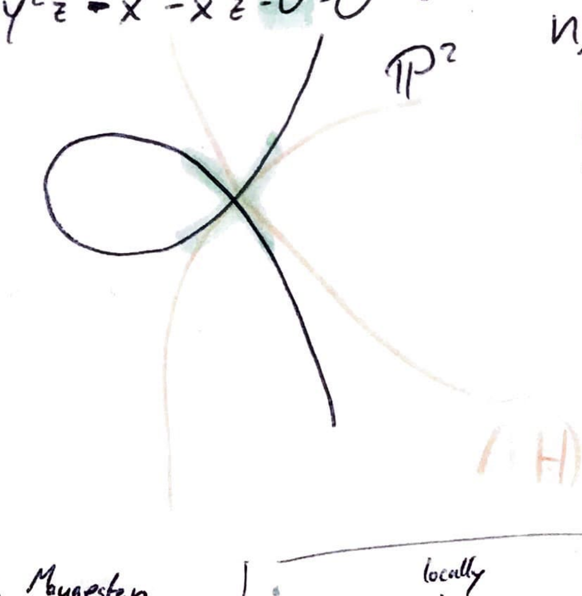
degree 3   degree $3 \cdot (3-2) = 3$

$= (D \cdot V(H)) = 9$

Bézout's thm

$D: f = y^2 z - x^3 - x^2 z - 0 = 0$

$\mathbb{P}^2$



$n_1 = \#\{\text{smooth } p \in D : (T_p \cdot C)_p = 3\}$

$= \sum_{(0,0,1) \neq p \in D \cap V(p)} (D \cdot V(H))_p + \underbrace{\cancel{(D \cdot V(H))_{(0,0,1)}}}_{=6}$

$= \cancel{9} \to 3$

for us $f = F(x,y,1)$, $g = H(x,y,1)$.

$$\boxed{(C \cdot C')_p \overset{\text{locally}}{=} (V(f) \cdot V(g))_{(0,0)} = \dim_{\mathbb{C}} \mathbb{C}[x,y]_{(x,y)} / (f,g)}$$

Refrences:

- Master thesis Paul A. Maugesten 'Sectantic points on plane alg. curves'
- 'Plane alg. curves', Gerd Fischer.

Question: What about higher degree curves maximally tangent to D?

Spoiler: $N_2 = \frac{21}{4} \boxed{= 3 \cdot \frac{3}{4} + 3}$

$3 = n_2$ ← count of smooth conics max. tangent to D.

Degree ≥ 2 & Gromov-Witten theory

$$\overline{\mathcal{M}}_{0,d}(\mathbb{P}^2 | D) = \{(C, f, p)$$

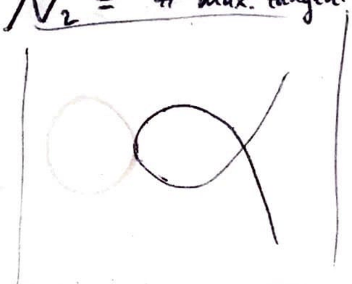
(virtual) dimension $= 0$

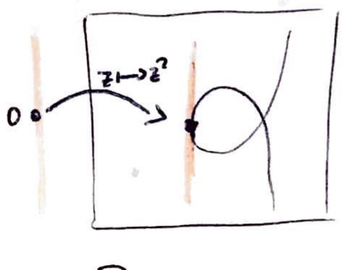
$$N_d := \# \overline{\mathcal{M}}_{0,d}(\mathbb{P}^2 | D)$$

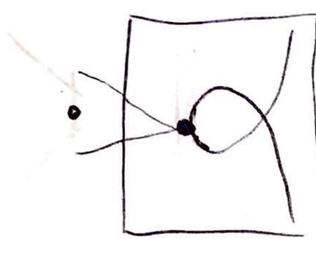
Pros: Deformation invariance, birat. inv., degenerations...

- $d=1$: $C = \mathbb{P}^1$, $f = \mathrm{id}$ onto image. $\Rightarrow N_1 = n_1 = 1$.
- $d=2$: $N_2 = \#$ smooth conics max. tangent to D $+ \; 3 \cdot \frac{3}{4} = \frac{21}{4}$

$\overset{!}{=} n_3$







$\Rightarrow n_2 = 3$

Prop: $N_d = \frac{3}{d^2}\binom{4d-1}{d-1}$ [van Garrel–Nabijou–S '23]

d	N_d		n_d
1	3	$=3$	3
2	$\frac{21}{4}$	$=3\cdot\frac{3}{4}+3$	3
3	$\frac{55}{3}$	$=3\cdot\frac{10}{9}+15$	15
4	$\frac{1365}{16}$	$=3\cdot\frac{35}{16}+3\cdot\frac{9}{4}+72$	72

GPS' : A smooth degree d' curve contributes

$$\frac{1}{(d/d')^2}\binom{(d/d')\cdot(3d'-1)-1}{(d/d')}$$

to N_d if $d' | d$.

Observation: $n_d \in \mathbb{Z}_{\geq 0}$.

Question: Geometric meaning of n_d?

Proof of Formula (idea)

- $R = \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \rightsquigarrow A = \mathrm{Aut}_{\mathbb{C}[[t]]}(R[[t]])$.

- given

for some $g(z,t)$

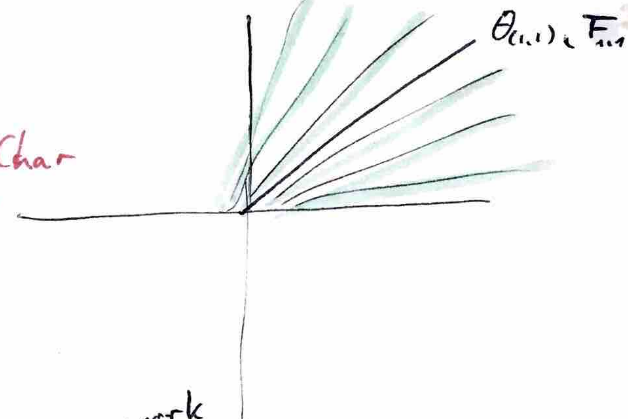
$$\underset{R[[t]]}{\underset{\cap}{f}} = 1 + t x^a y^b g(x^a y^b, t)$$

get $\Theta_{(a,b),f} \in A$ via

$$\Theta_{(a,b),f}(x) = x \cdot f^{-b}, \quad \Theta(y) = y \cdot f^a$$

- in general Θ's do not commute! E.g. $\overset{\curvearrowleft}{\prod_{a/b}} \Theta_{(a,b),F_{a,b}}$

$$\Theta^{-1}_{(0,1),(1+ty)^2} \circ \Theta_{(1,0),(1+tx)^2} \circ \Theta_{(0,1),(-)^2} \circ \Theta^{-1}_{(1,0)^2} = \Theta_{(1,1),1+t^2xy}$$



Reineke: $F^n_{a,b}$ encode Euler Char of moduli of framed quiver rep.^{ns} of $\bullet \rightrightarrows \bullet$.

GS: Mirror Sym.

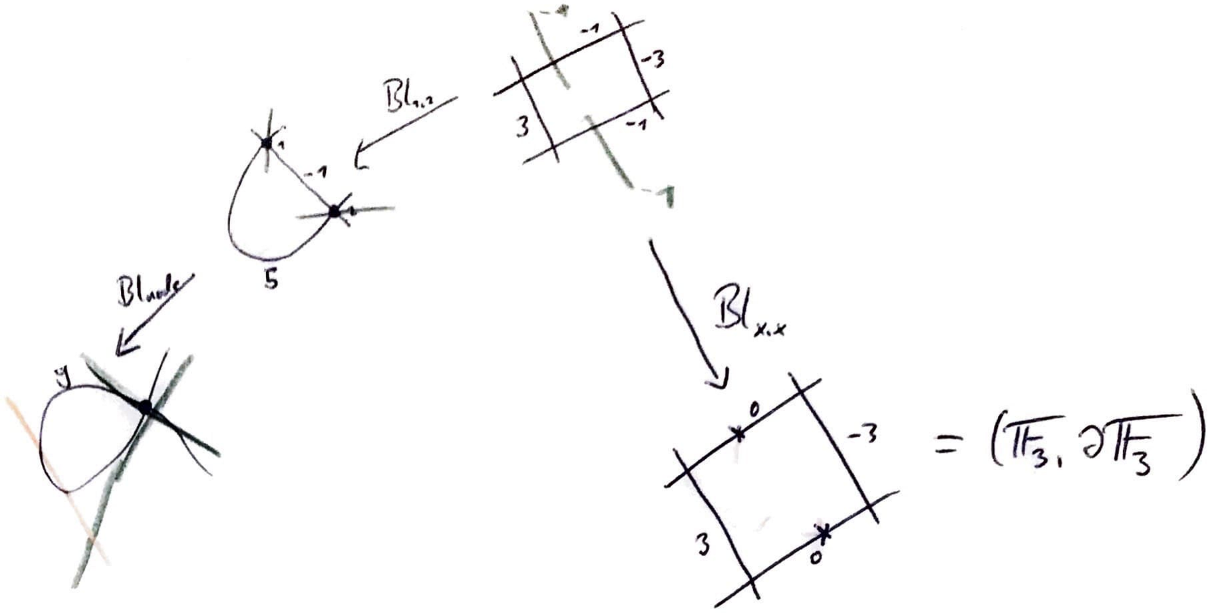
Gross-Pandharipande-Siebert '10 + some work: $F^3_{(1,1)} = \exp\left(\sum_{d>0} 3d \cdot N_d \cdot t^d x^d y^d\right)$

Reineke: Proves formula for $F^3_{(1,1)}$ => Formula for N_d.

Reference:

Gross-Pandharipande '0

'Quivers, Curves, trop. vertex



$$GW(\mathbb{P}^2 | D) = GW(\overline{\mathbb{P}}^2 | \overline{D}) \xleftrightarrow{\text{Combinatorics}} GW(\mathbb{F}_3, \partial\mathbb{F}_3)$$

birational invariance

$\|$

Scattering!