Gromov-Witten theory & the refined topological string

work in progress w/ Andrea Brini

$$GW_\beta(X) := \sum_{g\geq 0} u^{2g-2} \int_{[\overline{M}_g(X,\beta)]^{vir}} 1 \overset{X=K_{\mathbb{P}^2},\ \beta=L}{=} \frac{3}{2\sin\frac{u}{2}} = u^{-2}(\ ) + u^{0}(\ +\ ) + \dots$$

$\in u^{-2}\mathbb{Q}[[u]]$

BPS integrality

CY3 $\Rightarrow$ vdim $=0$

CY5 $\Rightarrow$ vdim $= 2-2g$ $\quad K_S \times \mathbb{C}^2$

equivariant GW(~~CY5~~)

GW(K_S) $\qquad$ GW($S\times\mathbb{C} \mid E\times\mathbb{C}$)

Setup

Let S smooth projective surface $/\mathbb{C}$.

$$T := (\mathbb{C}^\times)^2 \hookrightarrow (\mathbb{C}^\times)^3 \circlearrowright K_S \times \mathbb{C} \times \mathbb{C}$$

$$(t_1,t_2) \longmapsto (t_1^{-1}t_2^{-1}, t_1, t_2)$$

$$K_S \times \mathbb{C} \times \mathbb{C} \longrightarrow S$$

Rem: T fixes holom. 5-form.

defined via localisation

$$GW_\beta(K_S\times\mathbb{C}^2) := \sum_{g\geq 0} \text{"}\underbrace{\int_{[\overline{M}_g(K_S\times\mathbb{C}^2,\beta)]^{vir}_T} 1}_{\in H_T^{2g-2}(pt)^{loc}\,\cong\,\mathbb{Q}[\varepsilon_1,\varepsilon_2]^{loc}_{2g-2}}\text{"} \in (\varepsilon_1\varepsilon_2)^{-1}\cdot\mathbb{Q}[[\varepsilon_1,\varepsilon_2]]$$

if $M_g(K_S,\beta)$ proper!

$\mathbb{E} := \pi_*\omega_\pi$
$\pi : \mathcal{C}\to\overline{M}_g(-)$
$\mathbb{E}|_C = H^0(C,\omega_C)$

<u>Lem</u>: $GW_\beta(K_S\times\mathbb{C}^2)\Big|_{\varepsilon_1=-\varepsilon_2=iu} = GW_\beta(K_S)$

<u>Pf</u>: $$GW_\beta(K_S\times\mathbb{C}^2)\Big|_{\varepsilon_1=-\varepsilon_2=iu} = \sum_{g\geq 0}\int_{[\overline{M}_g(K_S,\beta)]^{vir}_T} (-\varepsilon_1\varepsilon_2)^{g-1}\, \underbrace{c_{-\varepsilon_1^{-1}}(\mathbb{E}^\vee)\, c_{-\varepsilon_2^{-1}}(\mathbb{E}^\vee)}_{=\,c_{-iu^{-1}}(\mathbb{E}^\vee\oplus\mathbb{E})\,=\,1}$$

total chern class evaluated at

Mumford '83 "Towards enum. geom. of moduli space of curves"

$$= \sum_{g\geq 0} u^{2g-2}\int_{[\overline{M}_g(K_S,\beta)]^{vir}_{\cancel{T}\to\{1\}}} 1 \qquad \square$$

This proves 1^{st} part of intro.

Now: BPS integrality.

BPS integrality

Assume S del Pezzo. Define

$$\mathrm{BPS}_\beta(S)(\varepsilon_1, \varepsilon_2) \in \mathbb{Q}[\![\varepsilon_1, \varepsilon_2]\!]$$

$$GW_\beta(K_S \times \mathbb{C}^2) =: \sum_{\substack{k \in \mathbb{Z}_{>0} \\ k \mid \beta}} \frac{1}{k} \frac{\mathrm{BPS}_{\beta/k}(S)(k\varepsilon_1, k\varepsilon_2)}{2\sinh\frac{k\varepsilon_1}{2} \cdot 2\sinh\frac{k\varepsilon_2}{2}}.$$

Conj: $\mathrm{BPS}_\beta(S) \in \mathbb{Z}[e^{\pm\varepsilon_1}, e^{\pm\varepsilon_2}]$.

Example: $\mathrm{BPS}_L(\mathbb{P}^2) = e^{+\varepsilon_1+\varepsilon_2} + 1 + e^{-\varepsilon_1-\varepsilon_2}$.

Rem: $\varepsilon_1 = -\varepsilon_2 = iu \rightsquigarrow \mathrm{BPS}_\beta(S)\big|_{\varepsilon_1=-\varepsilon_2=iu} =: \sum_{g \geq 0} \underset{\text{Gopakumar-Vafa invariant.}}{\overset{\mathbb{Z} \ni}{n_{g,\beta}}} \underbrace{\left(2\sin\frac{u}{2}\right)^{2g}}_{= e^{iu} - 2 + e^{-iu}}$

$\hookrightarrow$ Proven by Ionel-Parker and Doan-Ionel-Walpuski

Rem: $\mathrm{BPS}_\beta(S)$ studied in B-model picture by Choi-Huang-Katz-Klemm.

Rem: $(\mathbb{C}^\times)^2$ acts Calabi-Yau!

Rem: More involved if $\overline{M}_g(K_S, \beta)$ non-proper. Eg.

$$GW^{\boxed{\mathrm{red}}}_{k[\mathbb{P}^1]}(K_{\mathbb{P}^1} \times \mathbb{C}^3) \underset{\text{conj}}{=} -\frac{1}{k} \frac{1}{2\sinh\frac{\varepsilon_1+\varepsilon_2}{2} \cdot 2\sinh\frac{\varepsilon_1}{2} \cdot 2\sinh\frac{\varepsilon_2}{2}}$$

$\boxed{\text{Rem: Expected for ref. top. string}}$

Conj: $BPS_\beta(S) = \sum_{i,j\in\mathbb{Z}} (-1)^{i+j} q_+^i q_-^j \dim(\ \ )$

$q_\pm = e^{\frac{\epsilon_1 \pm \epsilon_2}{2}}$

Slogan: BPS = refinement of Euler char of M_β

refinement = perverse + Poincaré poly

$\pi : M_\beta \longrightarrow Chow_\beta(S)$

moduli space of 1dim Gieseker stable sheaves on S w/ support β & Euler char 1.

$$H^i\left({}^p\mathcal{H}^j R\pi_* \mathbb{Q}_{M_\beta}[\dim]\right) = gr_j^P H^i(M_{\beta,1}(S))$$

perverse filtration j-th graded piece

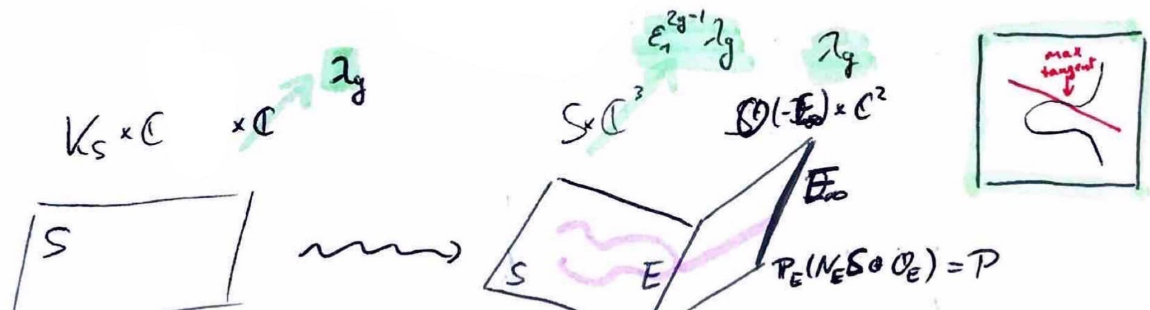
Thm: Conj. holds for $S=\mathbb{P}^2$ if $\epsilon_2=0$.

"Nekrasov-Shatashvili limit"

Prop: Let E be a smooth anti-canonical curve in S.

$$\epsilon_2 GW_\beta(K_S\times\mathbb{C}^2)\Big|_{\epsilon_2=0} = \frac{(-1)^{E\cdot\beta+1}}{E\cdot\beta}\sum_{g\geq 0}\epsilon_1^{2g-1}\int_{[\overline{M}_g^{max}(S/E,\beta)]^{vir}} \lambda_g .$$

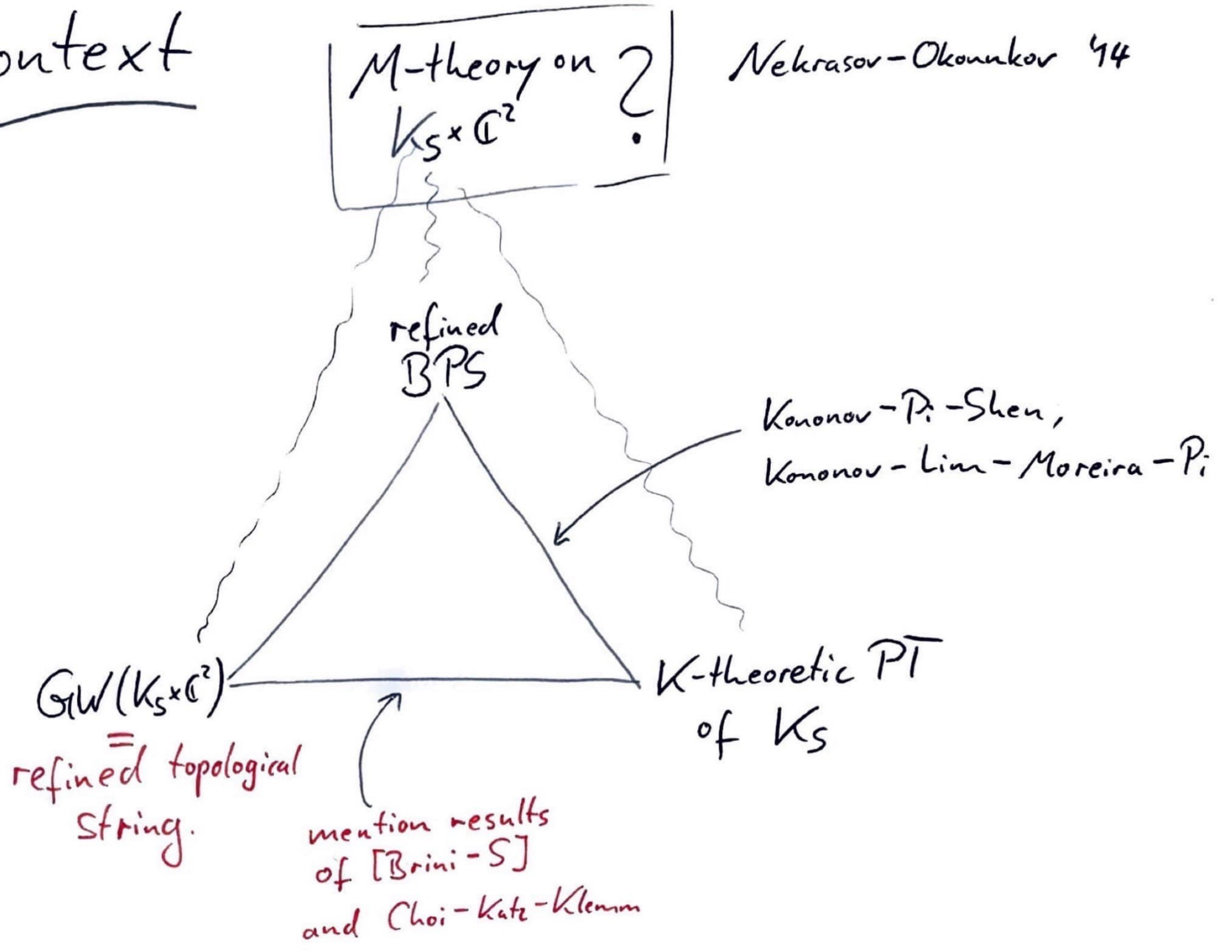
Pf:



$$GW(K_S\times\mathbb{C}^2) = \sum_{\substack{\text{splittings}\\(\Gamma_1,\Gamma_2)}} GW_{\Gamma_1}(S|E) * GW_{\Gamma_2}(P|E)$$

$\square$

Context



- Parts generalise to arbitrary CY3 with $\mathbb{C}^\times$-action.
- ... and Δ-part to CY5 w/ $\mathbb{C}^\times$-action.