Tangents to the Nodal Cubic

joint work with Michel van Garrel & Navid Nabijou
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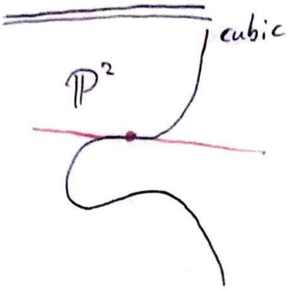
Motivation



$\mathrm{GW}_1(\mathbb{P}^2|D)$

$$n_1 := \#\{ p \in D : (T_p \cdot C)_p = 3 \}$$

$\det \mathrm{Hess}(f)_p = 0$ $\Updownarrow$

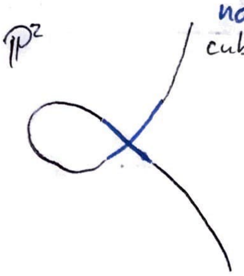
tangent line

$$= C \cdot V(\mathrm{Hess}(f))$$

degree 3 degree $3 \cdot (3-2) = 3$

$$= 9$$

$\rightsquigarrow$



$$n_1 := \#\{ \textbf{smooth } p \in D : (T_p \cdot C)_p = 3 \}$$

$$= 9 - \left(C \cdot V(\overset{=6}{\mathrm{Hess}(f)})\right)_{\text{node}}$$

$$= 3$$

<u>Question</u>: What about higher degree curves maximally tangent to D?

<u>Rem</u>: For D smooth this is solved by Gathmann (rat. curves) and Bousseau-Fan-Guo-Wu. (arbitrary genus)

<u>Spoiler</u>: $n_2 = \frac{21}{4}$.

$\rightsquigarrow$

$$\mathrm{GW}_2(\mathbb{P}^2|D) = 3 + 3 \cdot \frac{3}{4}$$

actually n_2

multicovering contribution of tangent lines

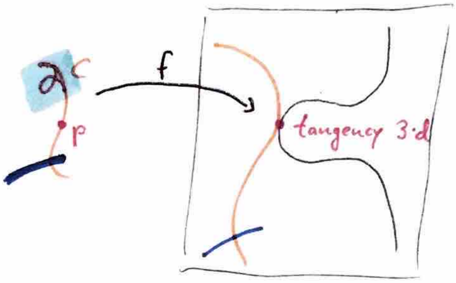
Main Result / Setup

We treat this as a moduli problem in the framework of GW-theory.

$$\overline{\mathcal{M}}_{d,g}(\mathbb{P}^2|D) = \text{"}\Big\{(C,f,p) \;\Big|\; C \ \text{~~smooth~~ rational} \overset{\text{genus } g}{} \text{ curve}, \ f: C \xrightarrow{} \mathbb{P}^2, \ f_*[C] \text{ degree}, \ C \text{ is maximally tangent to } D \text{ at } p \in C.\Big\}\text{"}$$

(log) stable map

proper!



↳ proper with (vir)dim = g. $\text{"} = [\overline{\mathcal{M}}_{d,g}(\mathbb{P}^2 \times \mathbb{A}^1, D \times \mathbb{A}^1)]^{vir}|_0 \text{"}$

↳ Define $GW_{d,g}(\mathbb{P}^2|D) := \deg (-1)^g \lambda_g \cap [\overline{\mathcal{M}}_{d,g}(\mathbb{P}^2|D)]^{vir} \in \mathbb{Q}$

Change n_1, n_2 to $GW_1(\mathbb{P}^2|D), GW_2(\mathbb{P}^2|D)$.

Prop: $GW_{d,0}(\mathbb{P}^2|D) = \frac{3}{d^2}\binom{4d-1}{d-1}$.

(via a [illegible] relation with the centra wall in the scatt. diagr. of the 3-Kronecker quiver + Reineke's formula)

Observation: $\frac{(-1)^{3d+1}}{3d} GW_{d,0}(\mathbb{P}^2|D) \overset{\text{direct comparison}}{=} GW_{d,0}^{T}\big(\underbrace{\mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)}_{= \mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^2}(-D)|_{\underbrace{\mathbb{P}^2\text{-node}}_{= \mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^1}(1)}}}\big)$

log = local

maximum contact w/ D ⟷ twist by $\mathcal{O}(-D)$

Now add g everywhere.

Thm: [vGN–] We have $\forall d>0$:

$$\frac{(-1)^{3d-1}}{2\sin\left(\frac{3d}{2}\cdot\hbar\right)}\left(\sum_{g\geq 0}\hbar^{2g-1}GW_{d,g}(\mathbb{P}^2|D)\left(\overset{(-1)^g}{\lambda_g}\right)\right) = \sum_{g\geq 0}\hbar^{2g-2}GW^T_{d,g}($$

Pf:

$$\Big\| \qquad\qquad \Big\| \text{ T-localisation}$$

$$\frac{(-1)^{3d-1}}{2\sin\left(\frac{3d}{2}\hbar\right)}\cdot\left(\sum_{g\geq 0}\hbar^{2g-1}GW_{\beta,g}\left(\underbrace{Bl_{node}\mathbb{P}^2}_{=\mathbb{F}_1}|D+E\right)\left((-1)^g\lambda_g\right)\right) \overset{\circledast}{=} \sum_{g\geq 0}\hbar^{2g-2}GW_{\beta,g}\left(Tot\,\mathcal{O}_{\mathbb{F}_1}(-D)|E\right)$$

Rmk: We actually prove this for arbitrary projective toric surfaces and D an irred. anti-canonical divisor with a node at a torus fixed point. and $D\cdot\beta>0$.

Eg. ($\mathbb{P}^1\times\mathbb{P}^1$ | nodal bisection), blowups...

holds without toric and anti-canonical assumption, i.e. for all bicyclic pairs st. the rhs is well-defined and we also allow stationary descendants.

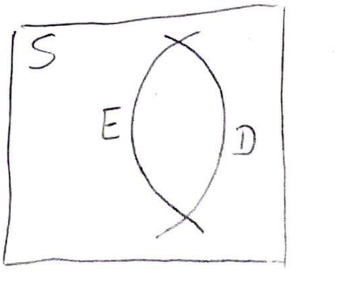
Why care? The right-hand side is fully solved by the topological vertex while lhs was so far unknown.

Cor: BPS integrality for $GW(\mathbb{P}^2|D)$, e.g. $\sum_{g\geq 0}\hbar^{2g-2}GW_{d,g}\left(\mathcal{O}_{\mathbb{P}^2}(-D)|_{\mathbb{P}^2\text{-node}}\right) \underset{\substack{\uparrow\\ d=1}}{=} \frac{1}{\left(2\sin\frac{\hbar}{2}\right)^2}$

Proof of $\circledast$

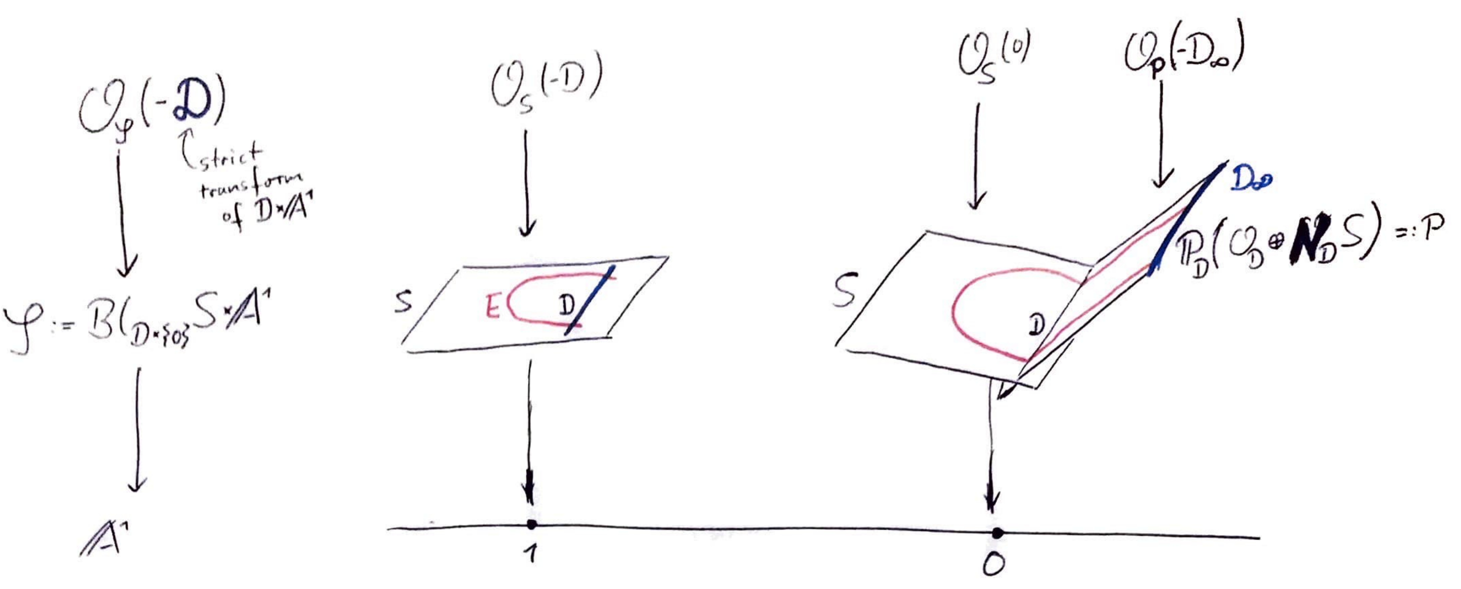
Key tool: Deformation to the normal cone
+
Degeneration formula in GW theory

Let $(S|D+E)$ be a bicyclic pair, which is:



Do deformation to the normal cone of D in S:

$$\mathcal{S} := Bl_{D\times\{0\}}S\times\mathbb{A}^1 \longrightarrow \mathbb{A}^1$$



$$GW_{\beta,g}(\mathcal{O}_S(-D) \mid E) = \sum_{\substack{\text{"splittings of curves"}\\(\Gamma_1,\Gamma_2)}} GW_{\Gamma_1}\big(\overset{=S\times\mathbb{A}^1}{\widetilde{\mathcal{O}_S(0)}} \mid D+E\big) \boxtimes GW_{\Gamma_2}\big(\mathcal{O}_P(-D_\infty) \mid D+E\big)$$

$$= \sum_{g_1+g_2=g} GW_{\beta,g_1}\big(\overset{\rightsquigarrow (-1)^{g_1}\lambda_{g_1}}{S\times\mathbb{A}^1} \mid D+E\big)\cdot(D\cdot\beta)\cdot GW_{(D\cdot\beta)\cdot\text{fibre},\,g_2}\big(\mathcal{O}_P(-D_\infty) \mid D+E\big)$$

$$+ \cancel{\sum_{\substack{\text{all other}\\\text{splittings}}} \cdots} \;\to = 0 \quad (\text{hard but possible to prove})$$

Note: $GW_{d\cdot\text{fibre},\,g_2}(\mathcal{O}_P(-D_\infty) \mid D+E) = GW_{d,g_2}\big(\underset{\text{fibre}}{\mathcal{O}_{\mathbb{P}^1}(0)} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \mid pt\big)$ ($\mathcal{O}_{\mathbb{P}^1}(-1)$: $N_{\text{fibre}}P$)

$$\overset{[\text{Bryan–Pandharipande}]}{=} \frac{(-1)^{d+1}}{d^2}\,[\hbar^{2g_1-1}]\left(2\sin\left(\frac{d}{2}\hbar\right)\right)^{-1}$$

Sum over g to get the result.

$\square$

Thm: For all bicyclic pairs $(S|D+E)$ with D rational and β a curve class with $D\cdot\beta>0$, we have

$$\frac{(-1)^{D\cdot\beta+1}}{2\sin\left(\frac{D\cdot\beta}{2}\hbar\right)}\left(\sum_{g\geq 0}\hbar^{2g-1}GW_{\beta,g,(\mathbf{C},\underset{\substack{\uparrow\\ \text{additional}\\ \text{maximum}\\ \text{contact}}}{(D\cdot\beta,0)})}(S|D+E)\left((-1)^g\lambda_g\prod_{i=1}^{n}\psi_i^{k_i}ev_i^*(\gamma_i)\right)\right)$$

$$=\sum_{g\geq 0}\hbar^{2g-2}GW_{\beta,g,\underset{\substack{\uparrow\\ \text{contact datum}\\ \text{with } E}}{\mathbf{C}}}(\mathcal{O}_S(-D)|E)\left(\prod_{i=1}^{n}\psi_i^{k_i}ev_i^*(\gamma_i)\right)$$

for stationary insertions.

either: γ_i is PD of point class;

or: $\gamma_i=1$ and $k_i=0$.