27.05.24, Fanosearch Seminar

Log vs. Open Gromov-Witten Theory

joint w/ van Garrel, Nabijou

Overview

Start with $(S|D)$:

- S projective rational surface $(+\dots)$
- $D \in |-K_S|$ s.n.c, singular

Conj: (Bousseau-Brini-van Garrel) Under technical conditions on $(S|D)$: $\exists (X|L)$ explicit!

$$(\dots)\cdot \underbrace{\mathrm{Log\,GW}(S|D)}_{\#\text{ curves}} = \underbrace{\mathrm{Open\,GW}(X|L)}_{\#\text{ curves (w/ boundary)}}$$

Thm: Holds if $D = D_1 + D_2$ has two components.

Applications

Log		Open
		[AKMV '05, LLLZ '09]
	$\xLeftarrow{①}$	toric ⇒ topological vertex method ⇒ (explicit) formulas
	$\xLeftarrow{②}$	BPS integrality [Yu '23]
tropical vertex [GPS '10] ⇒ formula $(\mathbb{P}^2 \mid \text{nodal cubic})$ genus = 0	$\xRightarrow{③}$	formula for $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n-2)$ genus = 0.

Comments: (skip unless relevant)

③ vGN- : $GW_{d[\mathbb{P}^1], g=0}\left(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n-2)\right) = \frac{(-1)^{nd-1}}{d^3} \binom{(n+1)^2 d - 1}{d-1}$

① $$\sum_{g \geq 0} \hbar^{2g-1} \, \mathrm{LogGW}_{g, d_0(H - \sum_i E_i) + \sum_i d_i E_i} \left(\begin{matrix} \text{del Pezzo} \\ \text{degree } 6 \end{matrix} \,\middle|\, \begin{matrix} D_1 = \text{line bl}_{\text{n}} \text{ up in a pt} \\ D_2 = \text{conic bl}_{\text{n}} \text{ up in 2 pts} \end{matrix} \right)$$

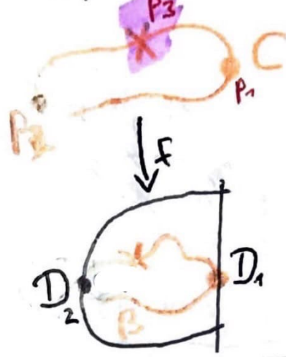
$$\underset{q = e^{i\hbar}}{=} \frac{[d_1]_q [d_2 + d_3]_q}{[d_0]_q [d_1 + d_2 + d_3 - d_0]_q} \begin{bmatrix} d_1 \\ d_0 - d_3 \end{bmatrix}_q \begin{bmatrix} d_1 \\ d_0 - d_2 \end{bmatrix}_q \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}_q \begin{bmatrix} \sum_i d_i - d_0 \\ d_4 \end{bmatrix}_q$$

$[a]_q := q^{a/2} - q^{-a/2}$

② $$\sum_{g \geq 0} \hbar^{2g-1} \, \mathrm{LogGW}_{g, \text{line}} \left(\mathbb{P}^2 \,\middle|\, \begin{matrix} \text{line} \\ + \\ \text{conic} \end{matrix} \right) \underset{\substack{\uparrow \\ q = e^{i\hbar}}}{=} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = \frac{q - q^{-1}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2}$$

Log GW

$(S \mid D_1 + D_2)$ as before $\rightsquigarrow \overline{M}_g^{\log}(S \mid D, \beta)$



$$\left\{ (C, f, p_1, p_2, p_3) \;\middle|\; \begin{array}{l} f: C \to S \text{ stable log map} \\ C \text{ genus } g \text{ curve,} \\ f_*[C] = \beta \\ p_1, p_2, p_3 \in C \\ f^* D_i = (D_i \cdot \beta)\, p_i \end{array} \right\} / \sim$$

if transv. intersection.

max. tangency

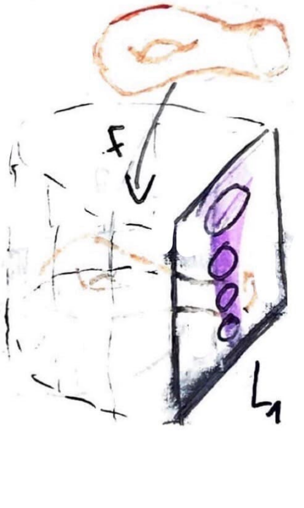
(exp) dim $(g-1) + 2 + 1 = g + 2$

λ_g $\quad ev_3^*(pt)$

$$\rightsquigarrow \mathrm{LogGW}_{g,\beta}(S \mid D_1 + D_2) := \int_{[\overline{M}_g^{\log}(S \mid D, \beta)]^{vir}} \lambda_g \, ev_3^*(pt)$$

Open GW

$(X \mid L_1)$ st. X toric CY3, $L \cong S^1 \times \mathbb{R}^2$ (Aganagic–Vafa) toric Lagrangian



$\rightsquigarrow \overline{M}_g^{open}(X \mid L_1, \boldsymbol{\beta})$

$$\left\{ (C, f, \partial C) \;\middle|\; \begin{array}{l} f: C \to X \text{ stable map} \\ C \text{ genus } g \text{ curve} \\ f_*[C] = \beta \\ \partial C \text{ winds } L_1\ (D_1 \cdot \beta)\text{-times} \end{array} \right\}$$

(exp) dim $= 0$

$$\rightsquigarrow \mathrm{OpenGW}_{g,\beta}(X \mid L_1) = \int_{[\overline{M}_g^{open}(X \mid L_1, \beta)]^{\bullet}} 1$$

Define $GW_\beta(*) := \sum_{g \geq 0} \hbar^{2g-2} \, GW_{g,\beta}(*)$.

includes $D_i \cdot \beta > 0$.

Thm: [vGN-, -] Under tech. cond. on $(S \mid D_1 + D_2)$

$\exists (X, L_1)$ st.

p_3 Automorphism factor

$$\mathrm{OpenGW}_\beta(X/L_1) = (-1)^{\cdots} \frac{1}{D_1 \cdot \beta} \cdot \boxed{\frac{\hbar}{2 \sin \frac{D_2 \cdot \beta \hbar}{2}}} \, \mathrm{LogGW}_\beta(S \mid D_1 + D_2)$$

irrelevant sign

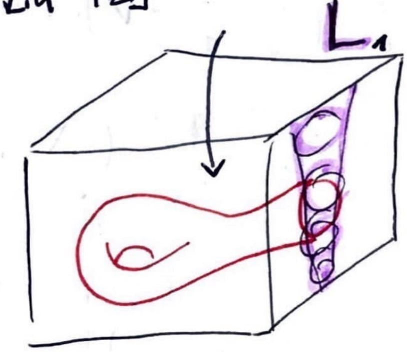
orientation of $\overline{\mathcal{M}}^{open}$

Proof idea: $\mathrm{Open\,GW}_\beta(X/L_1) \overset{①}{=} (\ldots) \cdot \mathrm{Log\,GW}_\beta(\overline{X} \mid \overline{D}_1)$

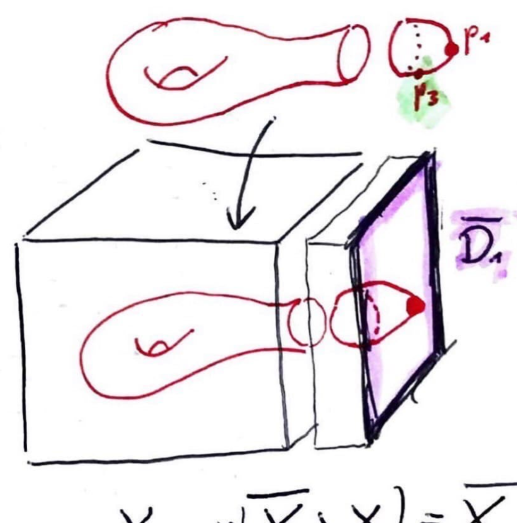
$\overset{②}{=} (\ldots) \cdot \mathrm{Log\,GW}_\beta(S \mid D_1 + D_2)$

Mention Construction of (X, L_1) here!

① [Fang-Lin '12] + extra work.



compactify X ⇝ & cap C



$X \rightsquigarrow X \cup (\overline{X} \setminus X) = \overline{X}$

$L_1 \rightsquigarrow \overline{D}_1$

Remark: • Need X toric here!

• Comment on the actual proof!

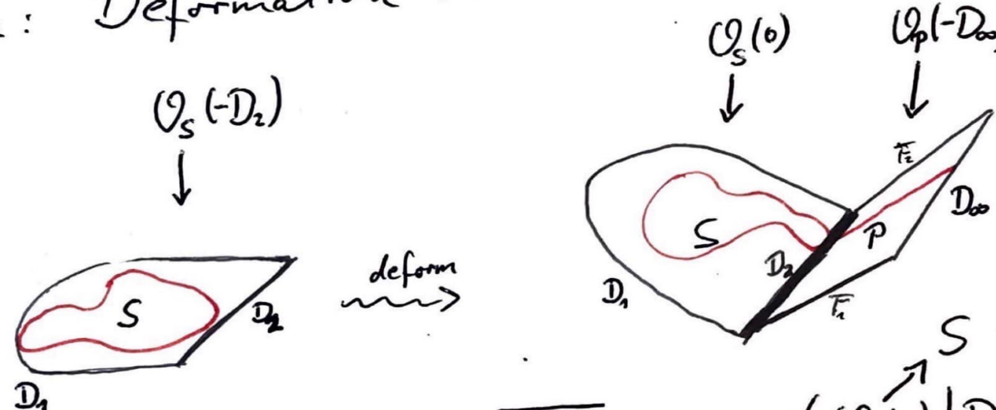
Construction: Given $(S\,|\,D_1+D_2)$

↳ Take $\bar{X} := \mathrm{Tot}\,\mathcal{O}_S(-D_2)$, $\bar{D}_1 := \pi^{-1}(D_1)$

↳ Take $X := \bar{X} - \bar{D}_1$, L_1 "near" $\bar{D}_1$.

⇝ need S, D_1 toric "up to deformation"!

② Idea: Deformation to the normal cone.



$$\mathrm{LogGW}(\mathcal{O}_S(-D_2)\,|\,\bar{D}_1) = \sum_{\substack{\text{splittings}\\(\Gamma_1,\Gamma_2)}} \overbrace{\mathrm{LogGW}_{\Gamma_1}(\mathcal{O}_S(0)\,|\,D_1+D_2)}^{S\,+\,\lambda_g} * \underbrace{\mathrm{LogGW}_{\Gamma_2}(\mathcal{O}_P(-D_\infty)\,|\,D_2+2F)}_{\boxed{\frac{(-1)^n}{2\sin\frac{\hbar(D_2\cdot\beta)}{2}}}}$$

□