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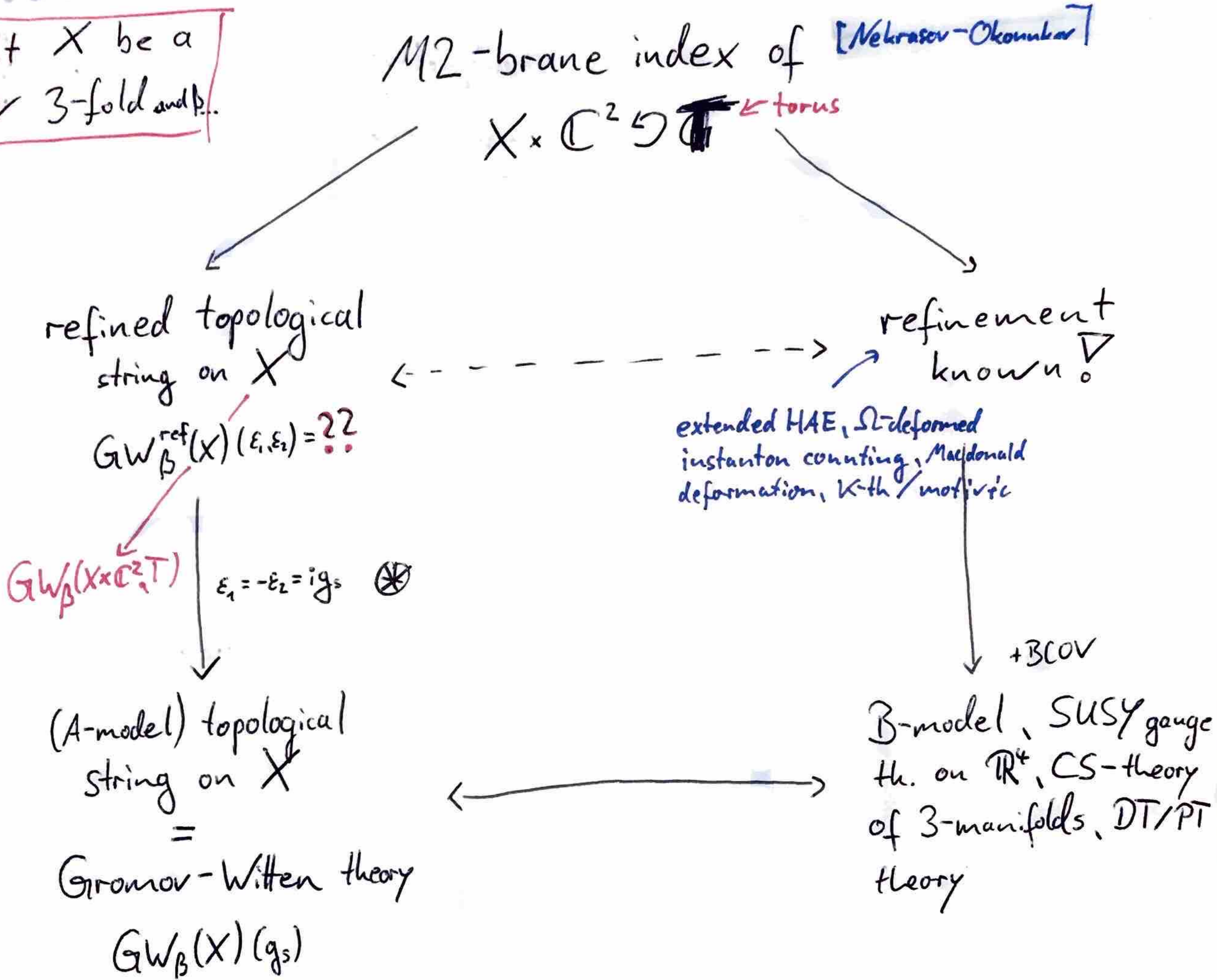
A worldsheet definition of the refined top. string

work in progress w/ Andrea Brini

Overview

Let X be a CY 3-fold and β...



M2-brane index of [Nekrasov-Okounkov]

$X \times \mathbb{C}^2 \circlearrowleft \mathbb{T}$ ← torus

refined topological string on X

$GW^{ref}_\beta(X)(\varepsilon_1,\varepsilon_2) = ??$

$GW_\beta(X \times \mathbb{C}^2, T)$

$\varepsilon_1 = -\varepsilon_2 = i g_s$ ⊛

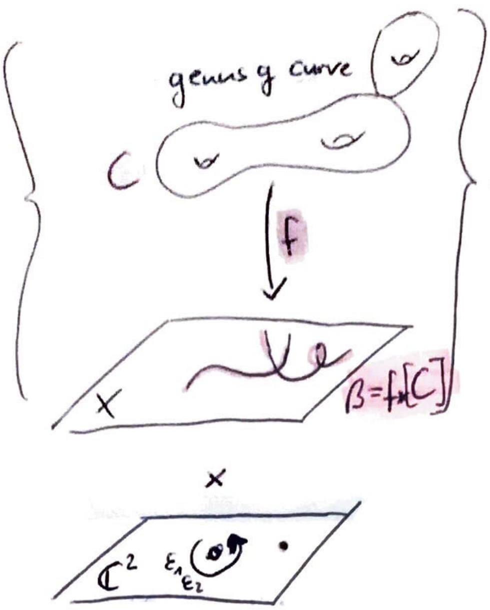
refinement known!

extended HAE, Ω-deformed instanton counting, Macdonald deformation, K-th / motivic

+BCOV

(A-model) topological string on X = Gromov-Witten theory $GW_\beta(X)(g_s)$

B-model, SUSY gauge th. on $\mathbb{R}^4$, CS-theory of 3-manifolds, DT/PT theory



$$= \overline{\mathcal{M}}_g(X \times \mathbb{C}^2, \beta)$$

$$GW_\beta(X) = \sum_{g \geq 0} g_s^{2g-2} \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{vir}} 1$$

$\uparrow$ vdim = 0 !

$$GW_\beta(X \times \mathbb{C}^2; T) := {}^{T}\!\!\int_{[\overline{\mathcal{M}}(X \times \mathbb{C}^2, \beta)]^{vir}} 1$$

dimension $\rightarrow$ $\updownarrow$ genus filtration

$$= \sum_{g \geq 0} {}^{T}\!\!\int_{[\overline{\mathcal{M}}_g(X \times \mathbb{C}^2, \beta)]^{vir}} 1 \qquad \overline{\mathcal{M}}_g(X,\beta) \times \mathbb{C}^2 \; /\!/$$

$\uparrow$ vdim = 2-2g

equivariance $\updownarrow$ volume regularisation ! $\rightarrow$

$$= \frac{1}{\varepsilon_1 \varepsilon_2} \int_{[\overline{\mathcal{M}}_0(X,\beta)]^{vir}} 1$$

$\leftarrow$ genus zero invariant unrefined

$$+ \sum_{g > 0} {}^{T}\!\!\int_{[\overline{\mathcal{M}}_g(\cancel{X},\beta)]^{vir}} \Lambda^\vee(\varepsilon_1)\, \Lambda^\vee(\varepsilon_2)$$

Assumption: $T \circlearrowright X \times \mathbb{C}^2$ satisfies

i) $\omega_{X \times \mathbb{C}^2}$ fixed

ii) ω_X not fixed $\Rightarrow$ X non-compact !

iii) extension of $\mathbb{C}^* \circlearrowright \mathbb{C}^2$ anti-diagonal

Lemma: ⊛ holds, i.e. $GW_\beta(X \times \mathbb{C}^2; T)\big|_{\varepsilon_1 = -\varepsilon_2 = i g_s} = GW_\beta(X).$

Remark:
- (iii) needed $\rightsquigarrow$ contravariance
- (ii) ensures refinement non-trivial
- (i) seems important in practise...

Remark: $GW_\beta(X \times \mathbb{C}^2, \beta) \in$ [illegible] $\mathbb{Q}[[\varepsilon_1, \varepsilon_2, u_3, u_4, \ldots, u_{\dim T}]]$.

Conj: (Rigidity) The generating series only depends on $\varepsilon_1, \varepsilon_2$ if $\overline{\mathcal{M}}_g(X,\beta)$ is compact $\forall g \geq 0$.

E.g. resolved conifold
local del Pezzo surf's

Rem: refined vertex. [Iqbal, Kozcaz, Vafa]

Evidence

Good news: Most unrefined techniques carry over!

From now on $Z = X \times \mathbb{C}^2$

Resolved conifold: $X = \mathrm{Tot}\ \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ modulo a technical assumption on T-action:

$$GW_{d[\mathbb{P}^1]}(Z,T) = \frac{1}{d}\frac{1}{2\sinh\frac{d\varepsilon_1}{2} \cdot 2\sinh\frac{d\varepsilon_2}{2}}$$

Local $\mathbb{P}^2$: $X = \mathrm{Tot}\ \omega_{\mathbb{P}^2}$, we show:

1) finite generation depth $3(g+k-1)$, weight zero quasi-modular function of $\Gamma_1(3)$.
2) extended HAE [Krefl-Walcher]
3) orbifold regularity asym behaviour at orbifold pt. ∞
4) conifold leading term conifold pt. $\frac{1}{27}$

= "direct integration" – "conifold gap"

↳ uniquely determines $GW_\beta(Z,T)$ [Huang-Kashani-Poor-Klemm]

similar to Lho-Pandharipande, Coates-Iritani.

↳ Cor: refined BPS integrality + match w/ K-th. PT theory degree $9/7$ mod stable maps of genus $g > 84$.

[Choi-Katz-Klemm]

[Kononov-Lim-Moreira Pi-Shen]

[Nekrasov-Okounkov]

BPS integrality

- Rem: Resolved Conifold.

Conj: If $\forall g\geq 0, \beta>0$ $\overline{M}_g(X,\beta)$ is compact, there exist $\Omega_\beta(q_1,q_2)\in\mathbb{Z}[q_1^{\pm 1},q_2^{\pm 1}]$ such that

$$GW_\beta(X\times\mathbb{C}^2,T)=\sum_{k\mid\beta}\frac{1}{k}\frac{\Omega_{\beta/k}(e^{k\varepsilon_1},e^{k\varepsilon_2})}{2\sinh(\frac{k\varepsilon_1}{2})\cdot 2\sinh(\frac{k\varepsilon_2}{2})}.$$

Example: $X=$ Resolved Conifold: $\Omega_\beta=\delta_{\beta,[\mathbb{P}^1]}$.

X satisfies BPS integrality for $\varepsilon_2=0$. Moreover:

Stress: Special for del Pezzo surfaces!

Thm: $X=K_{\mathbb{P}^2}$ satisfies

$$\Omega_\beta(q_1,1)=\sum_{i\in\mathbb{Z}}(-1)^i\, b_i(\underset{\text{pure 1-dim sheaves on }\mathbb{P}^2}{\underset{\uparrow}{M_\beta}})\, q_1^{\frac{i}{2}-\frac{\dim M_\beta}{2}}$$

Question: equivariant meaning?

Pf: degeneration + [Bousseau, Maulik-Shen] $\square$

[Conj: Thm generalises to arbitrary del Pezzo & when $\varepsilon_2\neq 0$ add Maulik-Toda's perverse grading.

The new

Speculation: For general CY5 Z with T-action satisfying (i):

$$GW_\beta(Z,T)=ch_T\left(\log\mathrm{Exp}\ \boxed{\text{M2-brane index}}\right)$$

[Nekrasov-Okounkov]

Question: How related to theorem?

Conservative speculation: $GW_\beta(Z,T)$ lifts to

$$K_T(pt)^{loc}_{\otimes\mathbb{Q}}\cong\mathbb{Q}\left[q_i^{\pm 1},\frac{1}{1-\prod_i q_i^{k_i}}\right]$$

Evidence: $\beta=1$, toric + β primitive

Example:
$$GW_\beta(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-2)\times\mathbb{C}^3,T)\underset{\underset{d\leq 3,\ g\leq 6}{\uparrow}}{\overset{\text{conj}}{=}}-\frac{1}{d}\frac{2\sinh\frac{d(\varepsilon_1+\varepsilon_2+\varepsilon_3)}{2}}{\prod_{i=1}^{3}2\sinh\frac{d\varepsilon_i}{2}}$$

Implications/Conjectures:

- rank r DT of X $\longleftrightarrow$ GW of $Z=X\times\widetilde{\mathbb{C}^2/\mu_r}$
- K-th PT = equiv. GW $\Rightarrow$ rigidity, vertex formalism
- formulas for $\overline{M}_{g,n}$-integrals