Lausanne 24.09.24

# Maps, Sheaves & Membranes

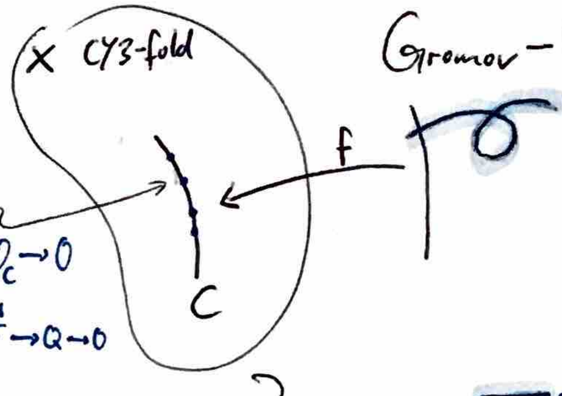
joint w/ A. Brini

Pandharipande–Thomas theory

- $\mathcal{O}_C \rightsquigarrow F$
- allow coker
- $P(X,\beta)$ birat. mod. of $\mathrm{Hilb}_\beta(X)$

$0 \to \mathcal{I}_C \to \underbrace{\mathcal{O}_X \to \mathcal{O}_C}_{} \to 0$

$\mathcal{O}_X \xrightarrow{s} F \to Q \to 0$




Gromov–Witten theory

- domain = nodal curve
- allow contracted comp.

$$\coprod_\chi P_\chi(X,\beta) \supset \left\{ \begin{array}{c} \text{smooth } C \hookrightarrow X \\ \text{of class } \beta \end{array} \right\} \subset \coprod_g \overline{\mathcal{M}}_g(X,\beta)$$

$\chi = 1 - \chi(\mathcal{O}_C)$ hol. Euler char

## MNOPT conjecture:

$$\underbrace{\sum_n (-q)^n \chi(P_n(X,\beta), \hat{\mathcal{O}}_{vir})}_{= Z^{PT}} \;\overset{=}{\underset{q=e^\varepsilon}{=}}\; \underbrace{\sum_g (-\varepsilon^2)^{g-1} \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{vir}} 1}_{= Z^{GW}}$$

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E.g. $X = \mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

$$Z^{PT}_{[\mathbb{P}^1]} = -q \underbrace{-2}_{= -\chi_{top}(\mathbb{P}^1)} q^2 \underbrace{-3}_{= +\chi_{top}(\mathrm{Sym}^2\mathbb{P}^1)} q^3 - \dots = -\frac{q}{(1-q)^2} = -\frac{1}{(q^{1/2}-q^{-1/2})^2}$$

$\| q = e^\varepsilon$

$$Z^{GW}_{[\mathbb{P}^1]} = -\varepsilon^{-2} + \underbrace{\frac{1}{12}}_{= 2 \cdot \frac{1}{24}} - \frac{\varepsilon^2}{240} + \dots = -\frac{1}{(2\sinh\frac{\varepsilon}{2})^2}$$

$2 = \chi_{top}(\mathbb{P}^1)$, $\frac{1}{24} = \int_{\overline{\mathcal{M}}_{1,1}} \psi_1$

physics expectation

Given torus $T' \circlearrowright X$.

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{c} \text{M2-brane index of} \\ X \times \mathbb{C}^2 \circlearrowleft T' \times \mathbb{C}^\times_q \\ \mathcal{Z}^{M2}_\beta \in \underbrace{K_{T' \times \mathbb{C}^\times_q}(pt)^{loc}}_{\cong K_{T'}(pt)[q^{\pm 1}]} \end{array} \arrow[dl, "q \to 0"'] \arrow[dr, "q \to 1"] & \\
\begin{array}{c} \text{K-th PT [Nekrasov–Okounkov]} \\ \mathcal{Z}^{PT}_\beta := \sum_n (-q)^n \chi_{T'}(P_n(X,\beta), \hat{\mathcal{O}}_{vir}) \\ \in K_{T'}(pt)((q)) \end{array} \arrow[rr, mapsto, "\text{CONJECTURE}", "\mathrm{ch}_T"'] \arrow[d, mapsto, "\kappa = 0"] & & \begin{array}{c} \mathcal{Z}^{GW}_\beta := {}^T\!\!\int_{[\overline{M}^\bullet(X \times \mathbb{C}^2, \beta)]^{vir}_T} 1 \\ \in H^*_T(pt)^{loc} \ni \varepsilon_1, \varepsilon_2 \end{array} \arrow[d, mapsto, "\varepsilon_1 = -\varepsilon_2 = \varepsilon"] \\
Z^{PT}_\beta(q) \arrow[rr, mapsto, "q \mapsto e^\varepsilon"'] & & Z^{GW}(\varepsilon)
\end{tikzcd}
$$

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Given torus $T' \circlearrowright X$

$\hookrightarrow$ $\kappa :=$ $T'$-character scaling $\omega_X^{-1} \cong \mathcal{O}_X$

$\hookrightarrow$ Let $T := T' \times \mathbb{C}^\times_q \circlearrowright X \times \underbrace{\mathbb{C} \times \mathbb{C}}_{= \kappa^{1/2} q + \kappa^{1/2} q^{-1}}$

$\hookrightarrow$ set $\varepsilon_1 = c_1^T(\kappa^{1/2} q)$, $\varepsilon_2 = c_1^T(\kappa^{1/2} q^{-1})$ $\in H^2_T(pt)$

Lem: $\mathcal{Z}^{GW}_\beta \big|_{\varepsilon_1 = -\varepsilon_2 = \varepsilon} = Z_{GW}$

Pf: Mumford '83

$$\mathcal{Z}^{GW}_\beta = \sum_{g \geq 0} \int_{[\overline{M}_g(X \times \mathbb{C}^2, \beta)]^{vir}} \Lambda(\varepsilon_1) \Lambda(\varepsilon_2)$$

$(-\varepsilon^2)^{g-1}$

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E.g. $X = \mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

$$\sum_{d > 0} \mathcal{Z}^{PT}_{d[\mathbb{P}^1]} Q^d = -q - \underbrace{[2]_\kappa}_{(-\kappa^{1/2})^{n-1} \chi_{-\kappa}(\mathrm{Sym}^{n-1}\mathbb{P}^1)} q^2 - [3]_\kappa q^3 - \dots = \overbrace{\frac{Q}{\left((\kappa^{1/2} q)^{1/2} - (\kappa^{1/2} q)^{-1/2}\right)\left((\kappa^{1/2} q^{-1})^{1/2} - (\kappa^{1/2} q^{-1})^{-1/2}\right)}}^{= Q \cdot \chi_T(\mathbb{C}^2, K^{1/2}_{\mathbb{C}^2}) = Q \mathcal{Z}^{M2}_{[\mathbb{P}^1]}}$$

Exp

$\downarrow \mathrm{ch}_T$

$$\sum_{d > 0} \mathcal{Z}^{GW}_{d[\mathbb{P}^1]} Q^d = \frac{Q}{2 \sinh \frac{\varepsilon_1}{2} \cdot 2 \sinh \frac{\varepsilon_2}{2}}$$

Exp

[BS] modulo technical assumption on T-action

## Speculation: For general CY5 Z5T:

$$Z^{GW}(Z,T) = ch_T \, \mathrm{Exp}\, \mathcal{F}^{M2}(Z,T)$$

Consequences:

- refined BPS/GV integrality for $Z^{GW}$
- rank-r $DT/X \longleftrightarrow Z^{GW}(X \times A_r)$

Consequences of Conj. $*$:

- rigidity for $Z^{GW}$
- vertex formalism

## Evidence when $X = K_{\mathbb{P}^2}$

We show that $Z^{GW}(K_{\mathbb{P}^2} \times \mathbb{C}^2)$ satisfies:

i) finite generation $\mathcal{F}_{k,g}$ is wt $=0$, depth $=3(g-k-1)$ quasi-modular function of $\Gamma_1(3)$

ii) extended HAE

iii) orbifold regularity $\mathcal{F}_{k,g} = \mathcal{O}(y^{-1})$

iv) conifold leading asymptotics $\mathcal{F}_{k,g} = *_{k,g} \cdot \Pi_{ref}^{2-2g-2k}\left(1 + \mathcal{O}(\Pi_{ref}^{1})\right)$

(i) – (iv) + "conifold gap" $*_{k,g}\Pi^{2-2g-2k} + 0 + \dots + \mathcal{O}(\Pi^0)$ specify $Z^{GW}(K_{\mathbb{P}^2} \times \mathbb{C}^2)$ uniquely!

[Huang–Kashani-Poor–Klemm]

**Cor:** For $X = K_{\mathbb{P}^2}$, Conj $*$ holds up to degree 7 modulo $g > 84$ stable maps and the "conifold gap".

[Choi–Katz–Klemm, Kononov–Lim–Moreira–Pi, Nekrasov–Ok...]

Define $\mathcal{F}_{k,g}$ via $\log \mathcal{Z}^{GW}(\varepsilon_1, \varepsilon_2, Q) =: \sum_{k,g \geq 0} (\varepsilon_1 + \varepsilon_2)^{2k} (\varepsilon_1 \varepsilon_2)^{g-1} \mathcal{F}_{k,g}(Q)$

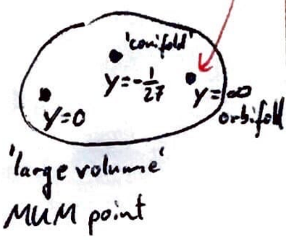
$\mathcal{F}_{0,0}(Q) \longleftrightarrow$ period of $\mathcal{Y}_y$

Family of elliptic curves w/ level 3 $\mathcal{Y}$

$\downarrow$

Kähler modulus $Q \xleftrightarrow[\text{map}]{\text{mirror}}$ cplx struct deformations $y \in \mathbb{P}(3,1)$

$Q(y) := e^{\Pi_\gamma(y)} = y\, e^{\varphi(y)}$




$K_{\mathbb{P}^2}$
$\downarrow$ c.r.
$\mathbb{C}^3/\mu_3$

Compactification of modular curve of $\Gamma_1(3)$ (level)

$[\mathbb{H}/\Gamma_1(3)]$

$\Gamma_1(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{matrix} a, d \equiv 1 \bmod 3 \\ c \equiv 0 \bmod 3 \end{matrix} \right\}$