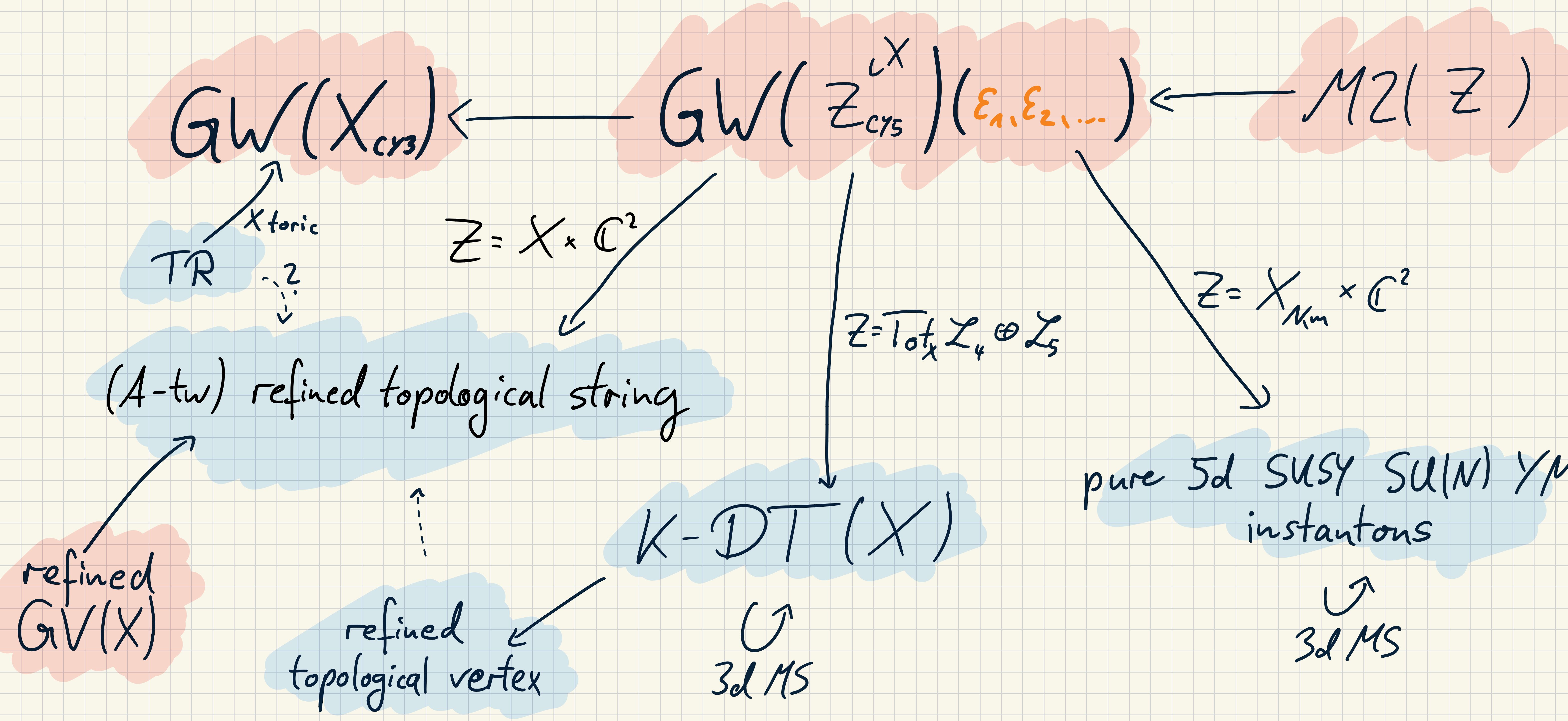
Membranes & Maps

ICTS

01/07/2026

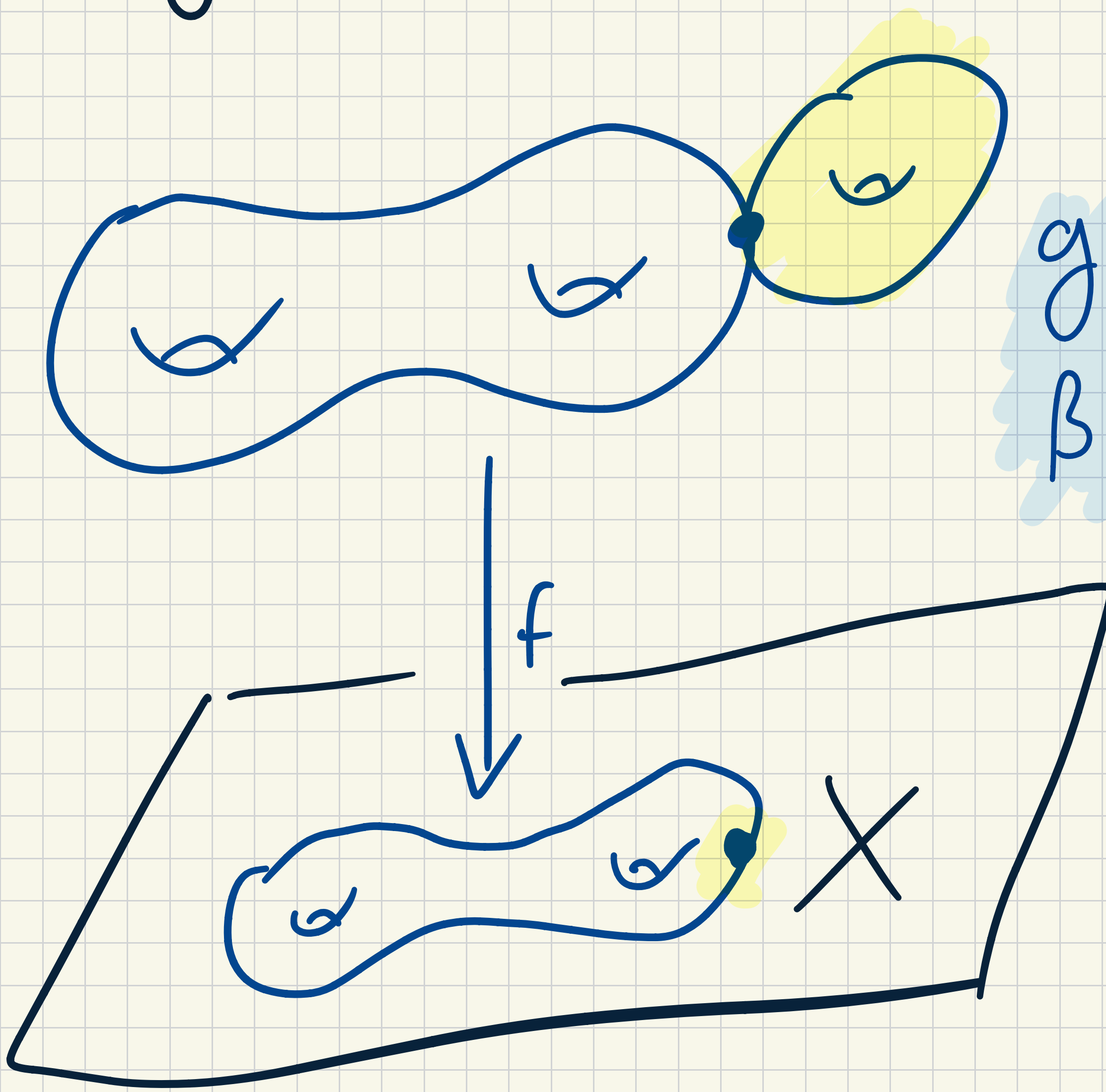
Overview



$GW(X_{CY3})$
$GW(Z^{\iota X}_{CY5})(\varepsilon_1, \varepsilon_2, \ldots)$
$M2(Z)$
TR
X toric
?
$Z = X \times \mathbb{C}^2$
(A-tw) refined topological string
$Z = \mathrm{Tot}_X \mathcal{L}_4 \oplus \mathcal{L}_5$
$Z = X_{N,m} \times \mathbb{C}^2$
refined GV(X)
refined topological vertex
K-DT(X)
3d MS
pure 5d SUSY SU(N) YM instantons
3d MS

Gromov-Witten theory

$$\overline{\mathcal{M}}_g(X,\beta) = \{(C,f)\}/\sim$$



$g = g_C$

$\beta = f_*[C]$

$X = CY3 \Rightarrow \operatorname{vdim} \overline{\mathcal{M}}_g(\,) = 0$

$$GW_\beta(X) := \sum_{g\geq 0} \varepsilon^{2g-2} \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{vir}} 1$$

$$\text{Ex: } GW_{\mathbb{P}^1}(X) = \varepsilon^{-2}\cdot 1 + \varepsilon^{0}\cdot \overset{g}{\frac{2}{24}} + \varepsilon^{2}\cdots$$

fix $\mathbb{P}^1 \subset X$ smooth

$$= \left(2\sin\tfrac{\varepsilon}{2}\right)^{-2}$$

[Faber-Pandharipande]

$$GW_C(X) = \left(2\sin\tfrac{\varepsilon}{2}\right)^{2g_C-2}$$

[Bryan-P]

GV conj/thm (Gopakumar–Vafa, Ionel–Parker, Doan–Ionel–Walpuski)

$$GW_\beta(X) = \sum_{\substack{k\beta'=\beta \\ k\in\mathbb{Z}_{>0}}} \sum_{g\geq 0} \frac{1}{k} \underset{\substack{\cap \\ \mathbb{Z}}}{n_{g,\beta'}} \left(2\sin\frac{ku}{2}\right)^{2g-2}$$

$\mathbb{Z}$ & zero for $g \gg 0$

$\Updownarrow$

Thm: $\exists\, \Omega_\beta(q) \in \frac{1}{(1-q)^2}\mathbb{Z}[q^{\pm 1}]$ such that under $q = e^{\sqrt{-1}\varepsilon}$:

$$GW_\beta(X) = \sum_{k\beta'=\beta} \frac{1}{k}\,\Omega_{\beta'}(q^k).$$

Ionel's talk $\rightsquigarrow$ real refinement

Next goal: equivariant refinement!

Equivariant refinement

$$X \rightsquigarrow \mathcal{M} \rightsquigarrow \int_{[\mathcal{M}]_T} \cdots \in \mathbb{Q}(\varepsilon_1, \ldots, \varepsilon_n)$$

$X \circlearrowleft T = (\mathbb{C}^\times)^n \ni (e^{\varepsilon_1}, \ldots, e^{\varepsilon_n})$

$\mathcal{M} \circlearrowleft T$

Example: $\displaystyle\int_{[\mathbb{C}]_{\mathbb{C}^\times_\varepsilon}} 1 = \frac{1}{\varepsilon}$

Defn: $\displaystyle GW_\beta(X, T) := \sum_{g \geq 0} \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{vir}_T} 1 \in \text{"}\widehat{\mathbb{Q}(\varepsilon_1 \ldots \varepsilon_n)}\text{"}$

Conj I Suppose Z is a CY3 (possibly non-cpct) with $Z \circlearrowleft T$ fixing $K_Z \cong \mathcal{O}_Z$.

$$\exists\, \Omega_\beta(q_i) \in \mathbb{Z}\left[\tfrac{1}{2}\right]\left[q_1^{\pm 1/2}, \dots, q_n^{\pm 1/2}, \frac{1}{1 - \prod_i q_i^{k_i/2}}\right]_{k \in \mathbb{Z}^n \setminus 0}$$

such that

$$GW_\beta(Z, T) = \sum_{k\beta' = \beta} \frac{1}{k}\, \Omega_{\beta'}(q_i^k) \quad \text{as } q_i = e^{\varepsilon_i}$$

"membrane index"

Next: ① Relation to GV conj

② Motivation

③ Evidence

Special case $\quad Z = X \times \mathbb{C} \times \mathbb{C}$, with $T \circlearrowright Z$ and $\mathbb{C}^\times_{\varepsilon_1} \circlearrowright \mathbb{C}$, $\mathbb{C}^\times_{\varepsilon_2} \circlearrowright \mathbb{C}$, $\quad T \longleftarrow \mathbb{C}^\times_{\varepsilon_1} \times \mathbb{C}^\times_{\varepsilon_2}$

REMEMBER!

Lem $\quad GW_\beta(X \times \mathbb{C}^2, T)\big|_{\varepsilon_1 = -\varepsilon_2 = \varepsilon} = GW_\beta(X)(\sqrt{-1}\,\varepsilon)$

Cor $\quad$ GV conj/thm $\not\Leftrightarrow$ Conj I for $X \times \mathbb{C}^2, T|_{\varepsilon_1 = -\varepsilon_2}$.

Conj II $\quad$ If $\overline{M}_g(X, \beta)$ proper $\forall g, \beta$ $\Rightarrow$ Experimental Observation: $GW_\beta(X \times \mathbb{C}^2, T)(\varepsilon_1, \varepsilon_2)$

$$\Omega_\beta(X \times \mathbb{C}^2, T) = \text{refined GV} \quad (\text{motivic, Betti, } \ldots)$$

[Holmes – S]

Rmk $\quad$ Assumption true for $K_{\mathbb{P}^2}$, $X_{N,m}$ $(|m| \leq N)$, ...

Thm [Brini – S] $\quad$ Conj II holds when $\varepsilon_2 = 0$ for $X = K_{\mathbb{P}^2}$.

Motivation/Speculation $\exists$ moduli of M2-branes M_β on Z

$$\Omega_\beta = \chi_T\big(M_\beta, \cancel{\text{coh. sheaf}}\big)$$
$$K_{M_\beta}^{1/2}\ ?$$

Rmk $K_{M_\beta}^{1/2}$ always exists in $K_T(M_\beta)^{\mathrm{loc}} \otimes \mathbb{Z}[\tfrac{1}{2}]$ $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots$

$$\Rightarrow \Omega_\beta \in \mathbb{Z}[\tfrac{1}{2}, q_i^{\pm 1}, \dots]$$

$K_{M_\beta}^{1/2} \in \mathrm{Pic}$ exists $\rightsquigarrow$ [Nekrasov-Okounkov]

$$\Rightarrow \Omega_\beta \in \mathbb{Z}[q_i^{\pm 1}, \dots].$$

Thm [S] Conj I holds w/ coeffs in $\mathbb{Z}$ if

i) $\# Z^T$, $\#\{1\text{-dim } T\text{-orbits}\} < \infty$

ii) $\forall p \in Z^T$: $T_p Z = q_p + q_p^{-1} + \ldots \in \mathrm{Rep}(T)$.

iii) technical condition on β

Key Observation (i - iii) $\Rightarrow$ $GW = \sum_{\text{partitions}} (\ldots) \prod_{p \in Z^T} \mathrm{TopVertex}(q_p)$.

OG un-refined vertex!

Non-Ex K_{dP}, $X_{N,m}$ beyond self-dual limit

Ex $Z = \mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3}$, $\mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2}$, ...

Conj [Giacchetto–Pandharipande–S] Conj I holds for local curves

$$Z = \mathrm{Tot}_C\, \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \oplus \mathcal{L}_5 \quad \circlearrowleft T$$

when $\beta = [C]$ with

$$GW_{[C]}(Z, T) = \Omega_{[C]}$$

$$= \chi_T\left(\overline{\mathcal{M}}_{g_C}(Z, [C]),\ K^{1/2}\right)$$

$$\overset{C = \mathbb{P}^1}{=} \prod_{i=2}^{5} \prod_{j=0}^{\deg \mathcal{L}_i} \left(2 \sinh \frac{\varepsilon_i - j\varepsilon_1}{2}\right)^{-1}$$