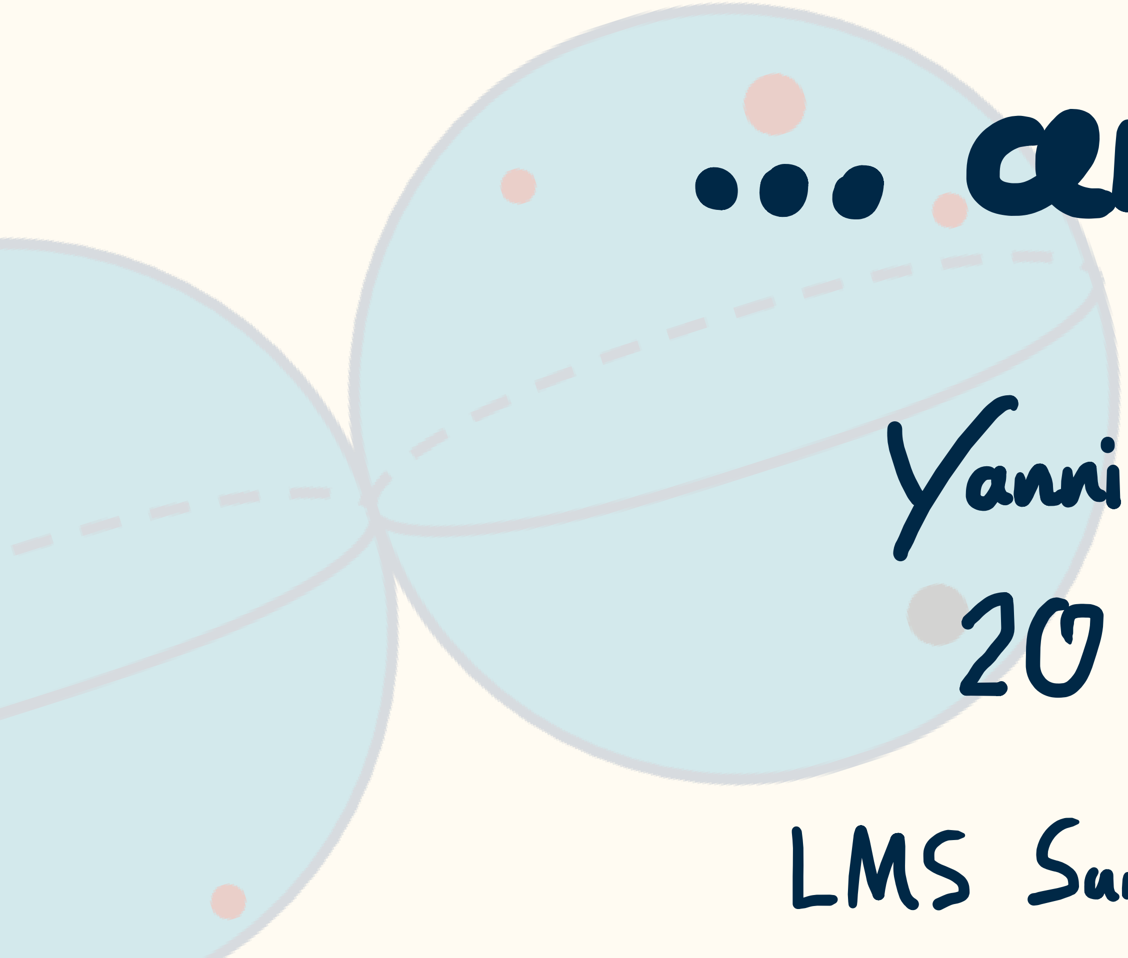
From Curves to Strings ... and back

Yannik Schüler

20 July 2023

LMS Summer School 2023



$\mathbb{P}^2$

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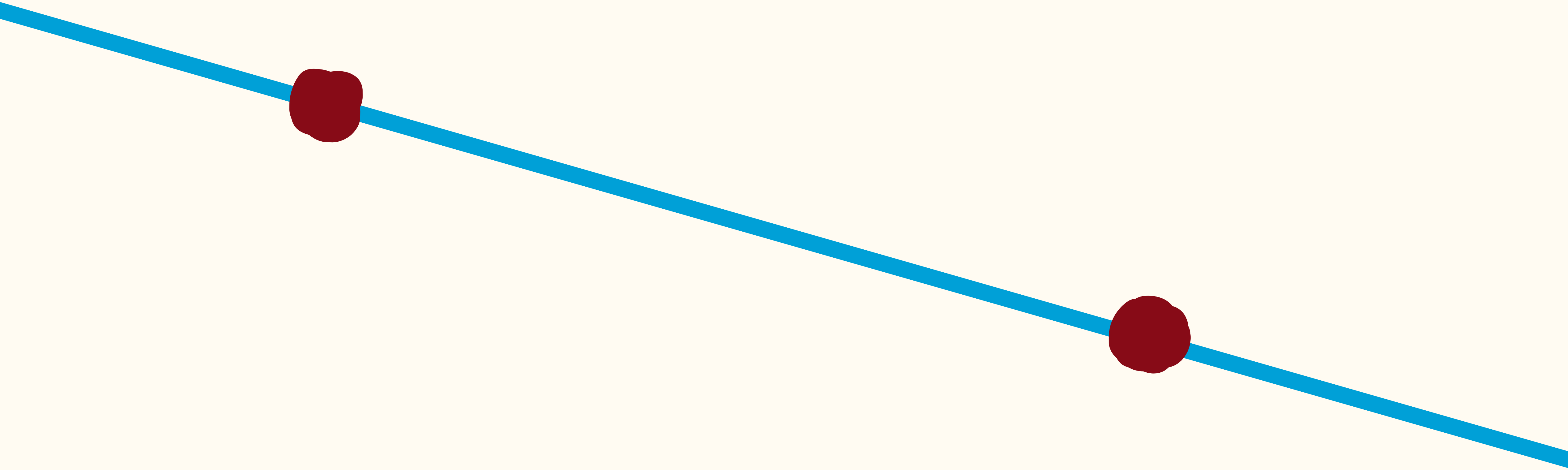
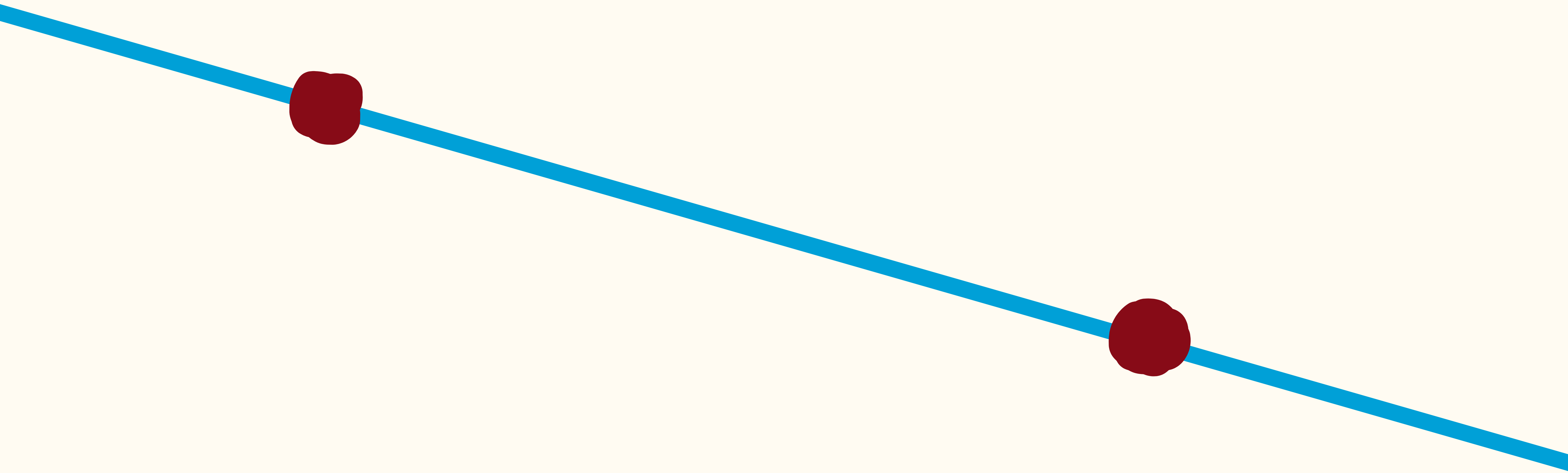


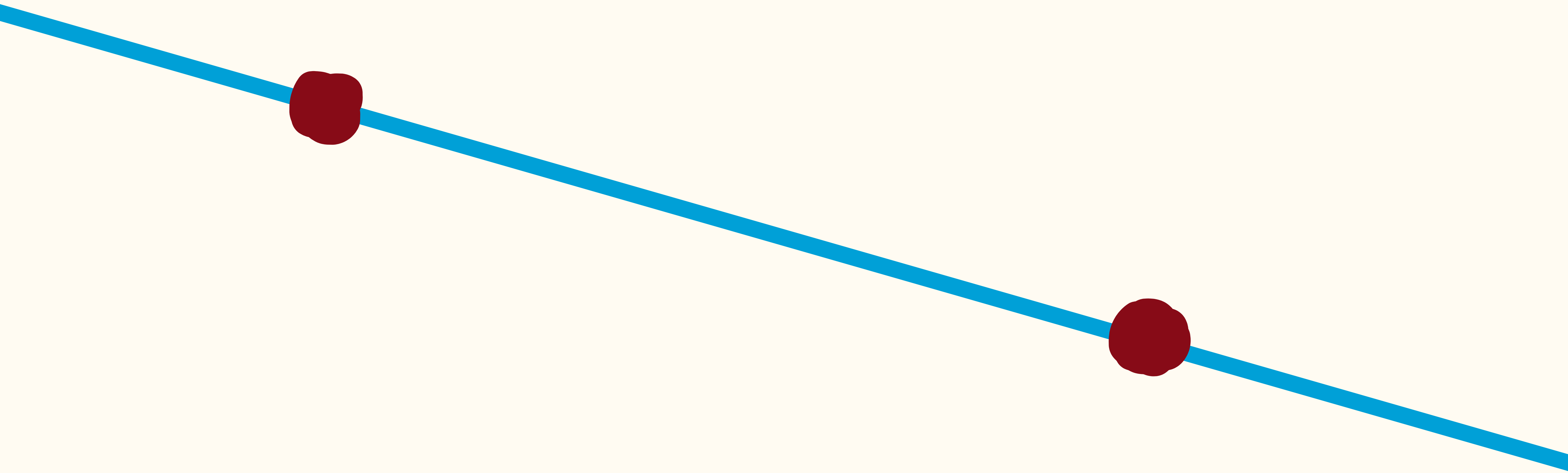
$\mathbb{P}^2$

$$\mathcal{L} = \{a + bx + cy = 0\}$$



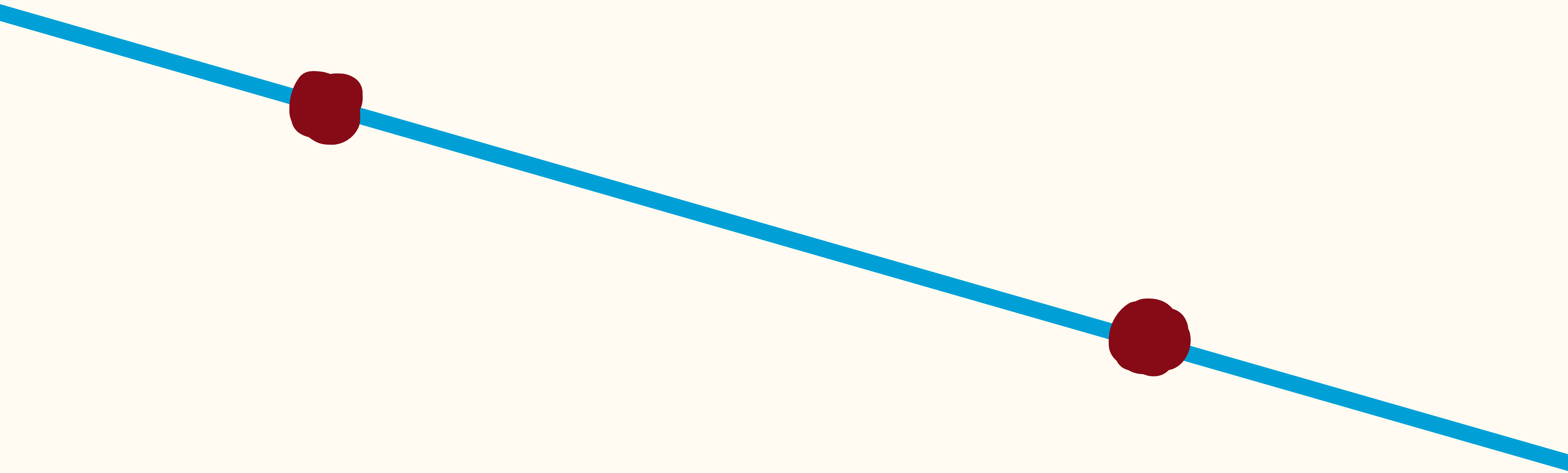
$\mathbb{P}^2$

$$\mathcal{L} = \left\{ \frac{a}{c} + \frac{b}{c}x + y = 0 \right\}$$



$\mathbb{P}^2$

$$\mathcal{L} = \{ a' + b'x + y = 0 \}$$



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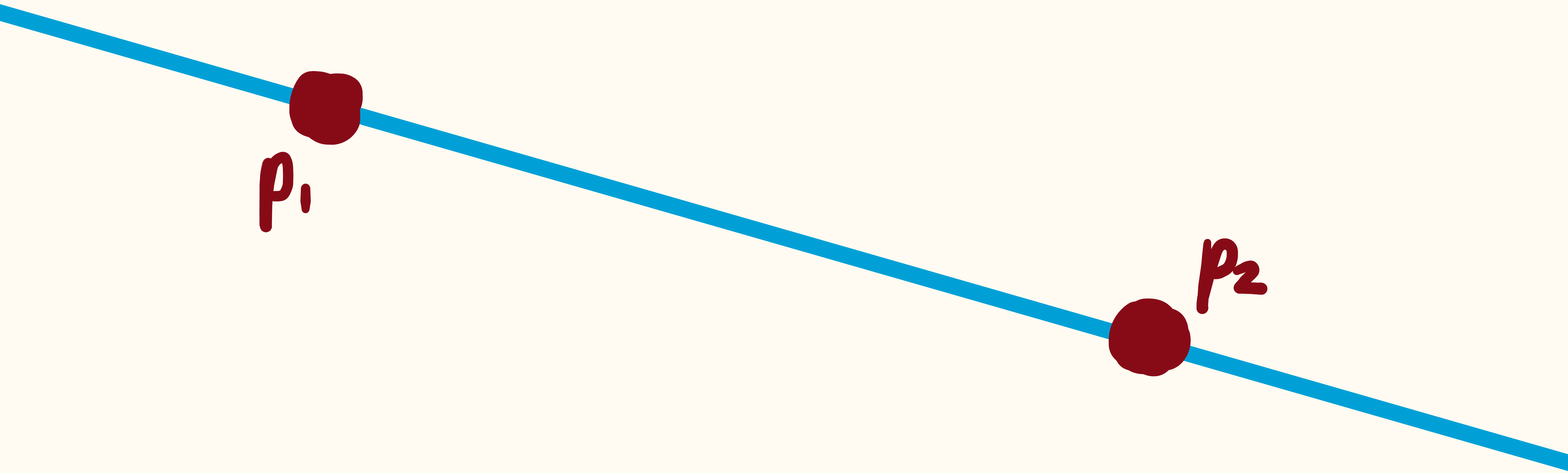

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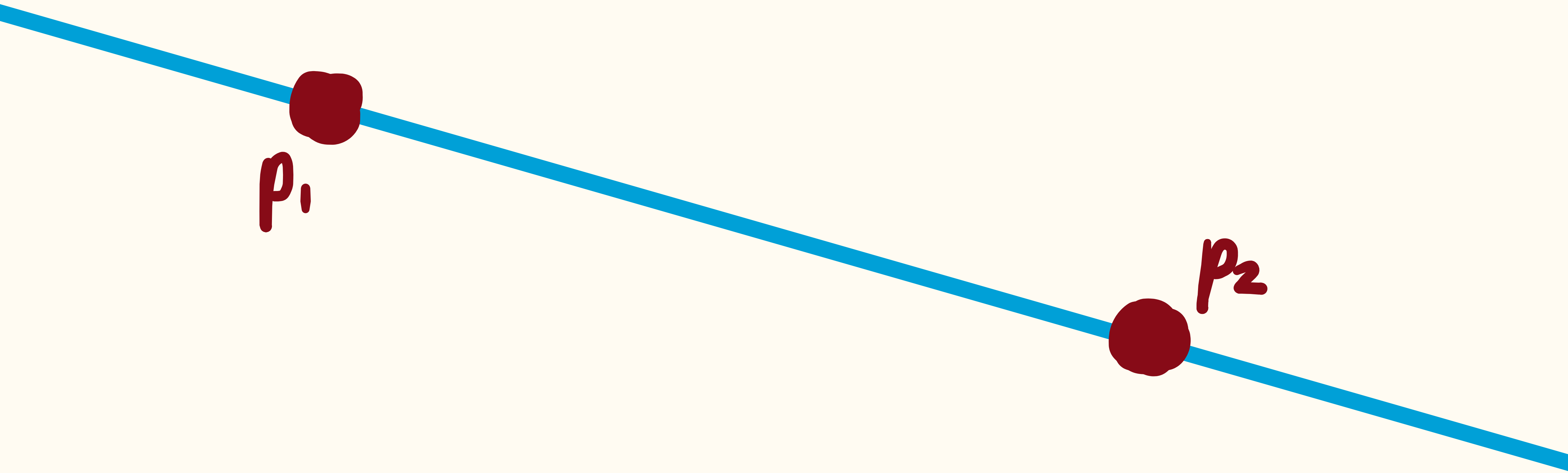
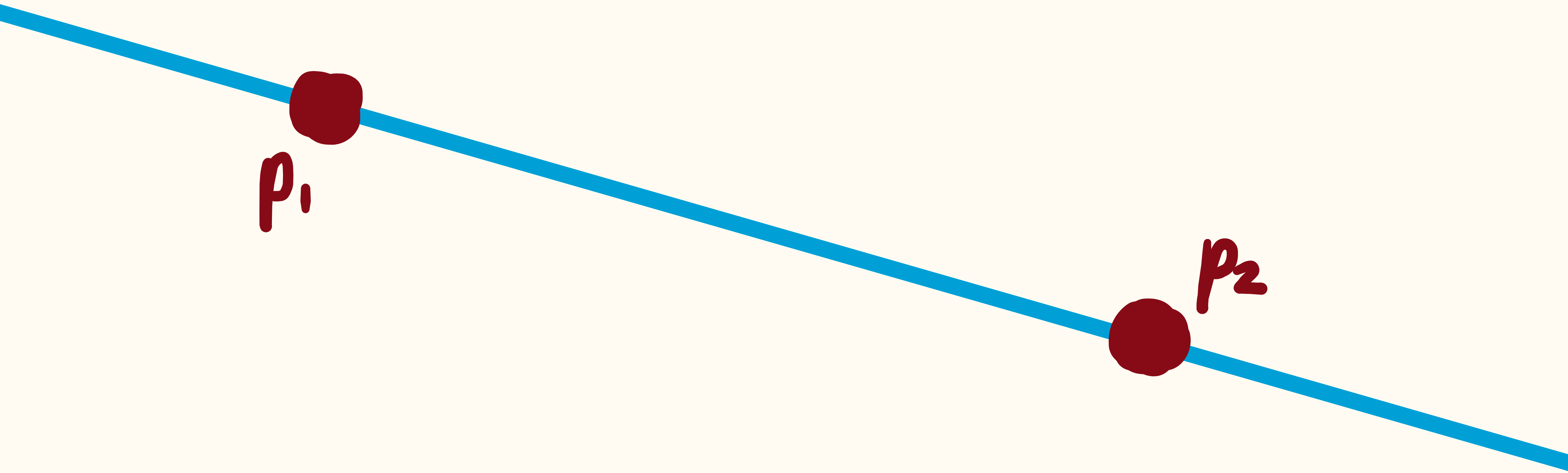


$$N_1 = \#\{\mathcal{L} \mid p_1, p_2 \in \mathcal{L}\}$$

$\mathbb{P}^2$

$$\mathcal{L} = \{ a' + b'x + y = 0 \}$$



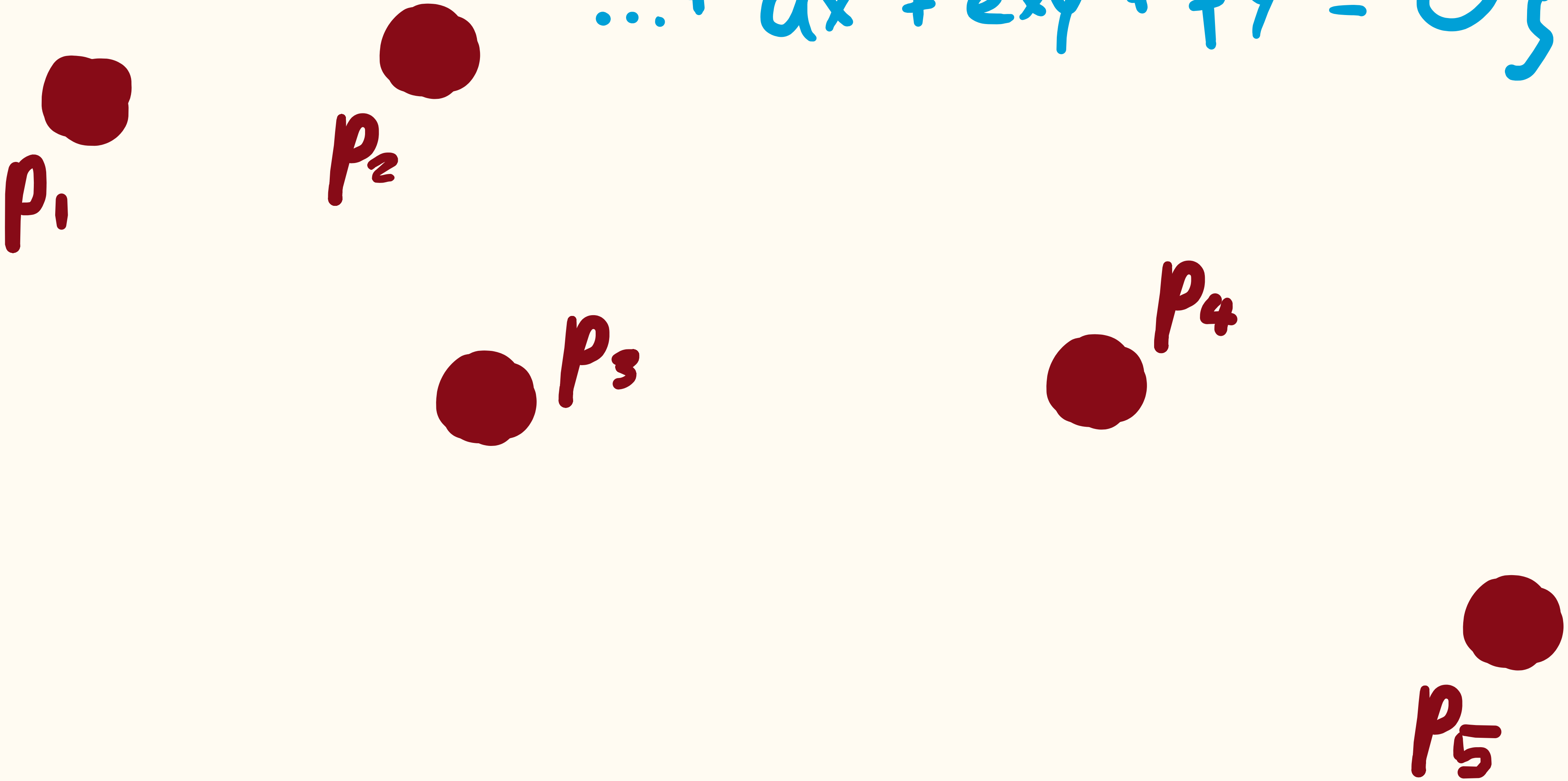
$$N_1 = \#\{ \mathcal{L} \mid p_1, p_2 \in \mathcal{L} \} = 1$$

$\mathbb{P}^2$

$$\mathcal{L} = \{ a + bx + cy + \dots + dx^2 + exy + fy^2 = 0 \}$$

$\mathbb{P}^2$

$$\mathcal{L} = \{ a + bx + cy + \dots + dx^2 + exy + fy^2 = 0 \}$$



$\mathbb{P}^2$

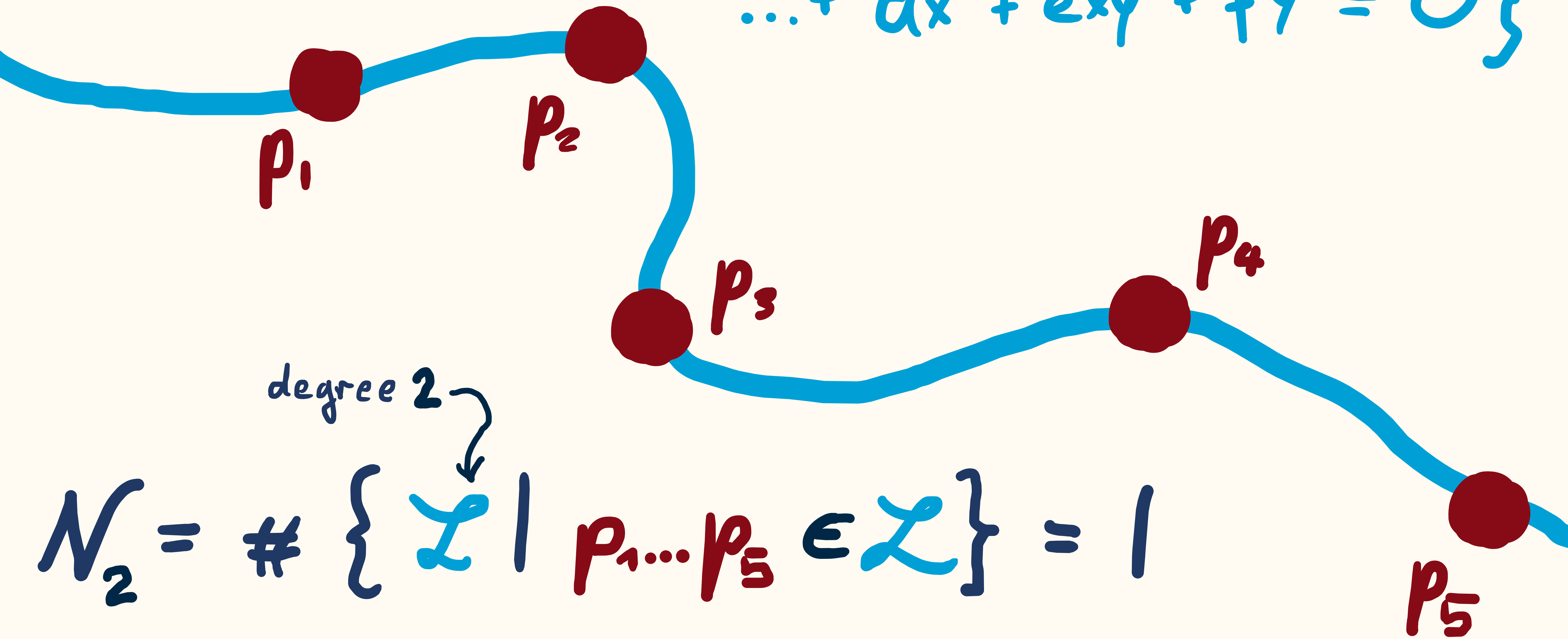
$$\mathcal{L} = \{ a + bx + cy + \ldots + dx^2 + exy + fy^2 = 0 \}$$

degree 2

$$N_2 = \# \{ \mathcal{L} \mid p_1 \ldots p_5 \in \mathcal{L} \}$$

$\mathbb{P}^2$

$$\mathcal{L} = \{a + bx + cy + \dots + dx^2 + exy + fy^2 = 0\}$$



degree 2

$$N_2 = \#\{\mathcal{L} \mid p_1 \dots p_5 \in \mathcal{L}\} = 1$$

$$N_d = \#\left\{ \begin{array}{l} \text{rational degree } d\ \mathcal{L} \\ \text{such that } p_1 \ldots p_{3d-1} \in \mathcal{L} \end{array} \right\}$$

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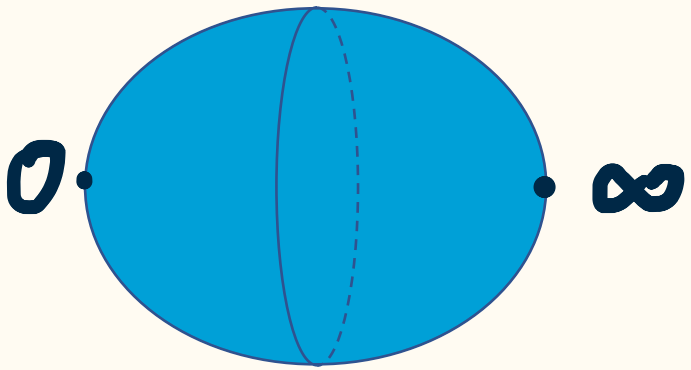
rational means $\mathcal{L} \cong \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$



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d	1	2	3	4	5	≥ 6
N_d	1	1				

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d	1	2	3	4	5	>6
N_d	1	1				

antiquity

$$N_d = \#\left\{\begin{array}{l}\text{rational degree } d\ \mathcal{L} \\ \text{such that } p_1 \ldots p_{3d-1} \in \mathcal{L}\end{array}\right\}$$

d	1	2	3	4	5	≥ 6
N_d	1	1	12			

antiquity (d = 1, 2)

1853 (d = 3)

$$N_d = \#\left\{\begin{array}{l}\text{rational degree } d \ \mathcal{L} \\ \text{such that } p_1 \ldots p_{3d-1} \in \mathcal{L}\end{array}\right\}$$

d	1	2	3	4	5	>6
N_d	1	1	12	620		
	antiquity	antiquity	1853	1873		

$$N_d = \#\left\{\begin{array}{l}\text{rational degree } d\ \mathcal{L} \\ \text{such that } p_1 \ldots p_{3d-1} \in \mathcal{L}\end{array}\right\}$$

d	1	2	3	4	5	≥ 6
N_d	1	1	12	620	87304	
	antiquity		1853	1873	~1980	

$$N_d = \#\left\{\begin{array}{l}\text{rational degree } d\ \mathcal{L} \\ \text{such that } p_1 \ldots p_{3d-1} \in \mathcal{L}\end{array}\right\}$$

d	1	2	3	4	5	≥ 6
N_d	1	1	12	620	87304	Recursion
	antiquity		1853	1873	~1980	Kontsevich ~1994

From Curves

From Curves

↓

to Strings

String Theory

Physics

Classical Mechanics

Quantum Mechanics

Quantum Field Theory

String Theory

Physics

Classical Mechanics

Quantum Mechanics

Quantum Field Theory

String Theory

Mathematics

Physics

Classical Mechanics

Quantum Mechanics

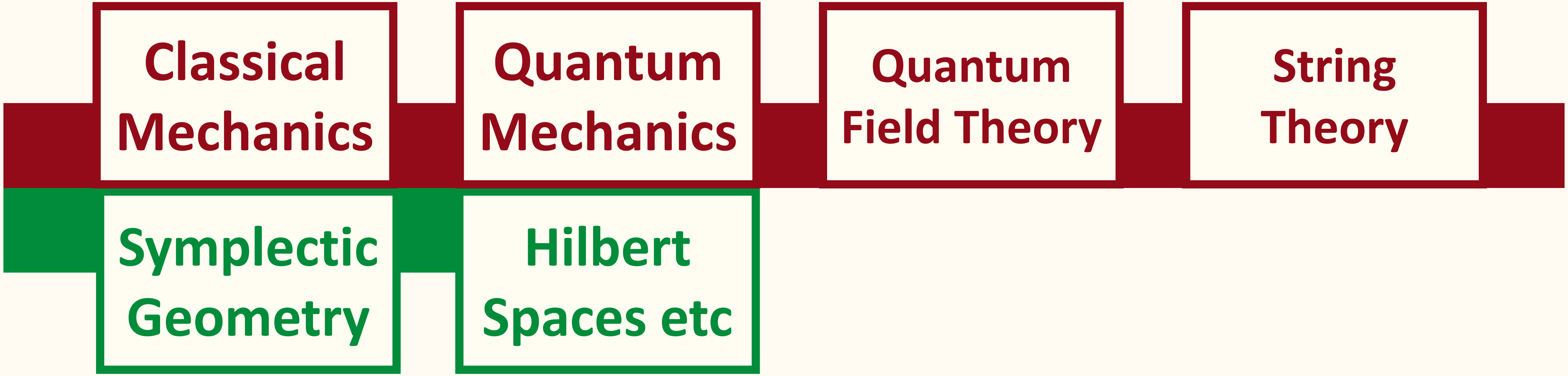
Quantum Field Theory

String Theory

Symplectic Geometry

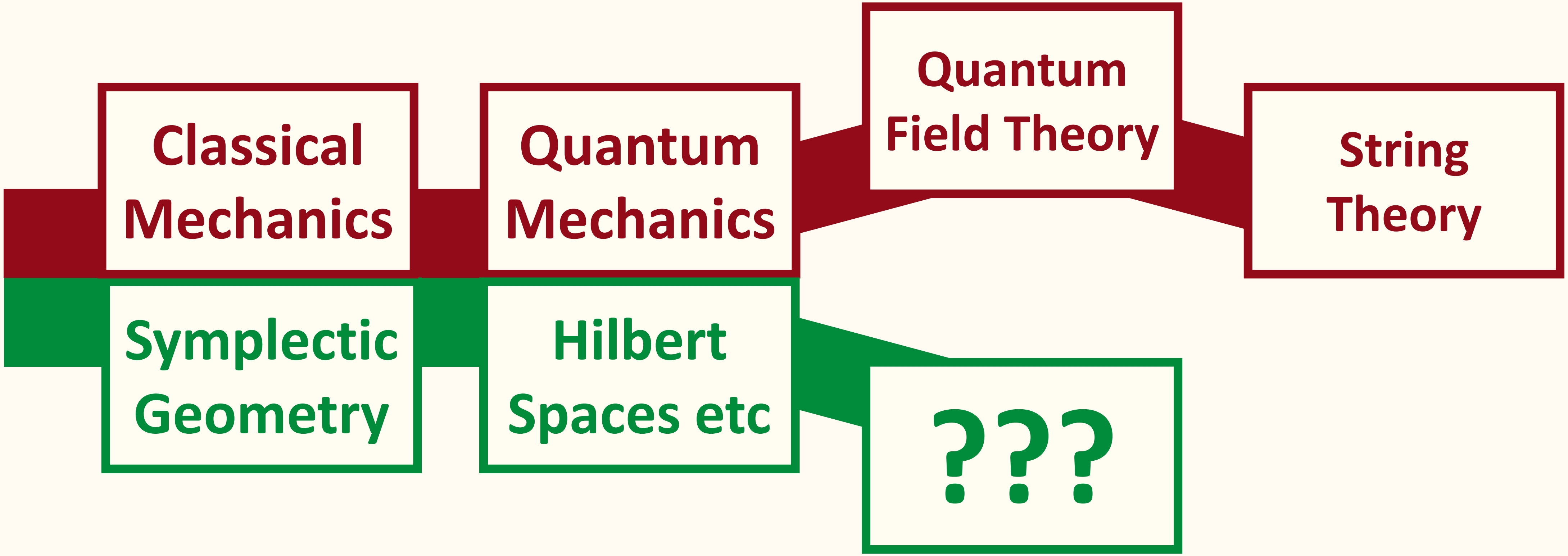
Mathematics

Physics



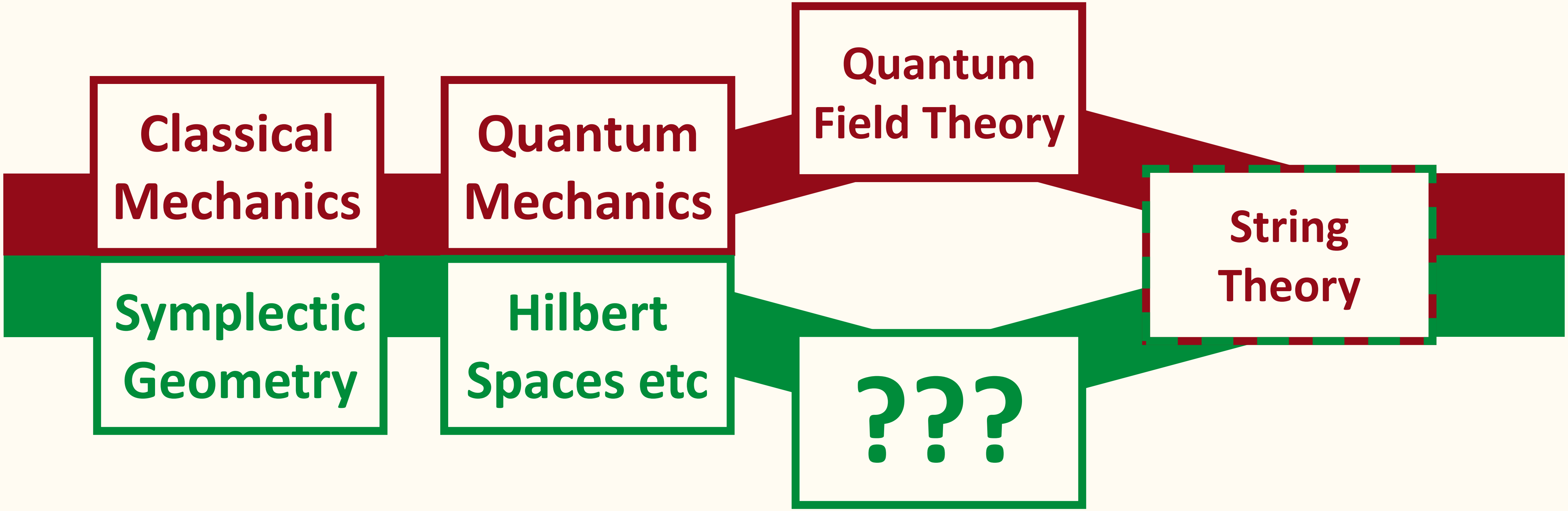
Mathematics

Physics

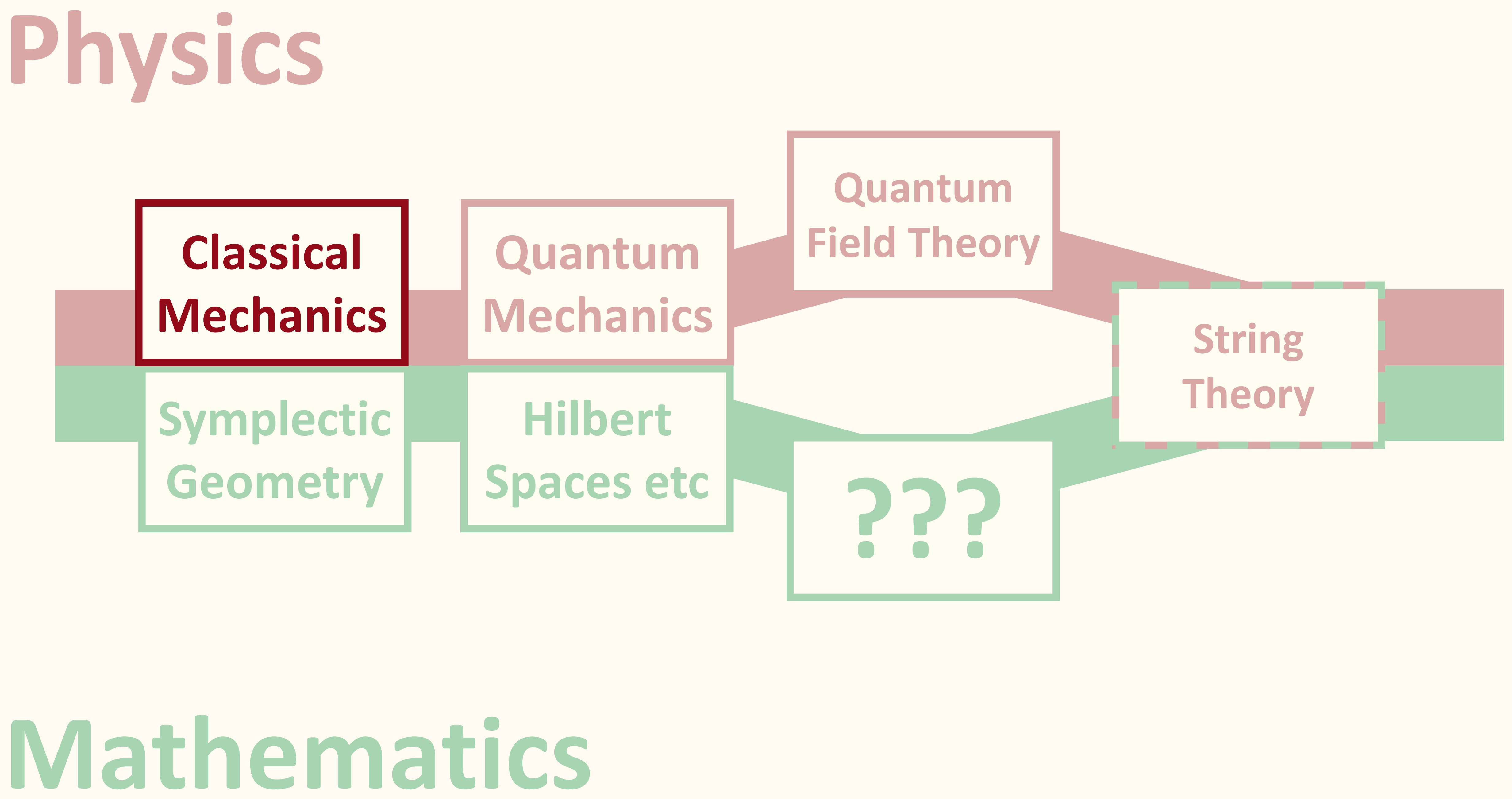


Mathematics

Physics

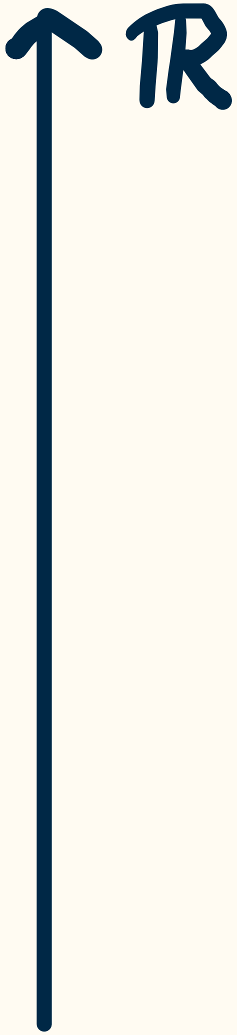


Mathematics

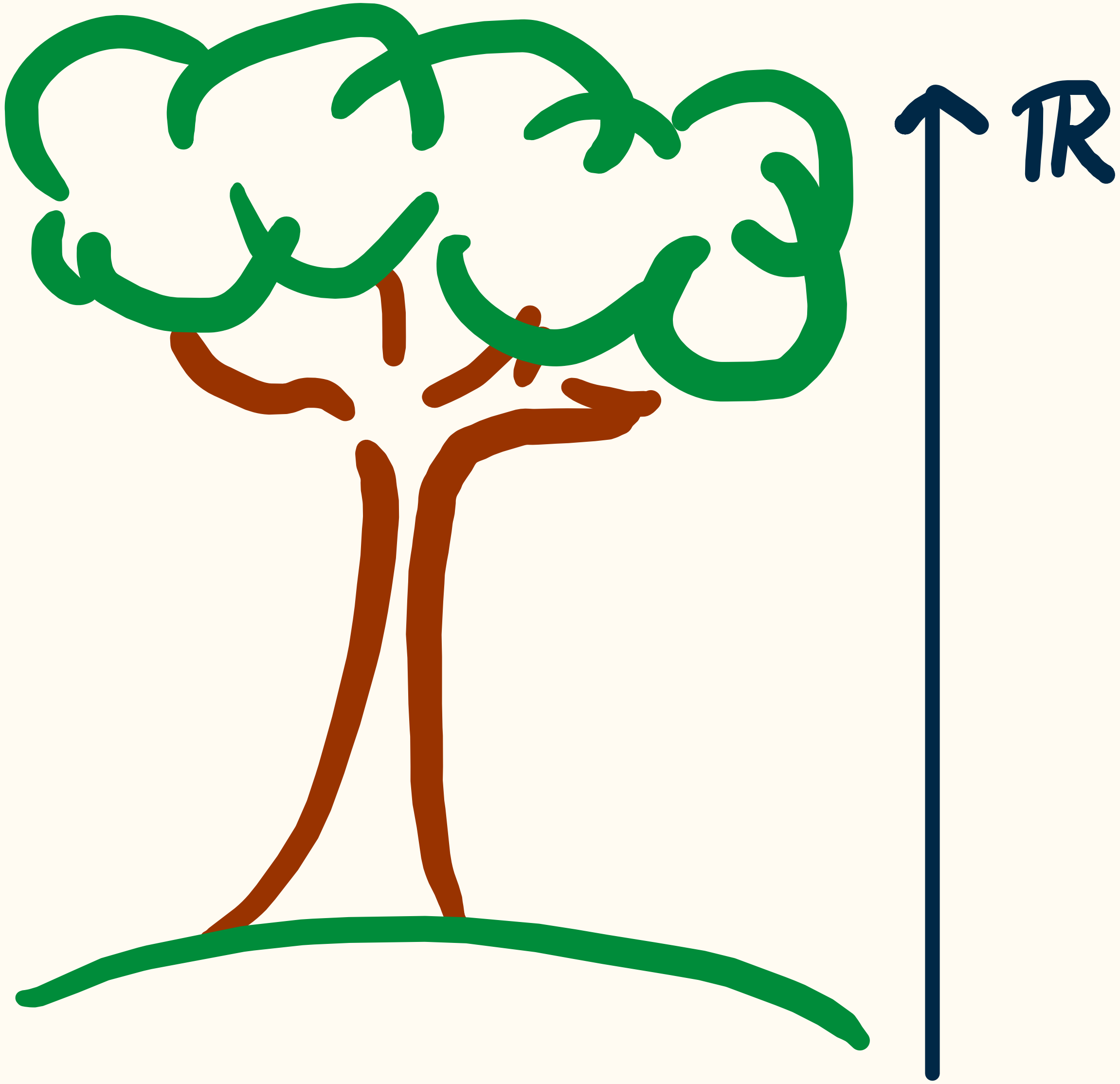

Physics
Classical Mechanics
Quantum Mechanics
Quantum Field Theory
String Theory
Symplectic Geometry
Hilbert Spaces etc
???
Mathematics

Classical Mechanics

Classical Mechanics

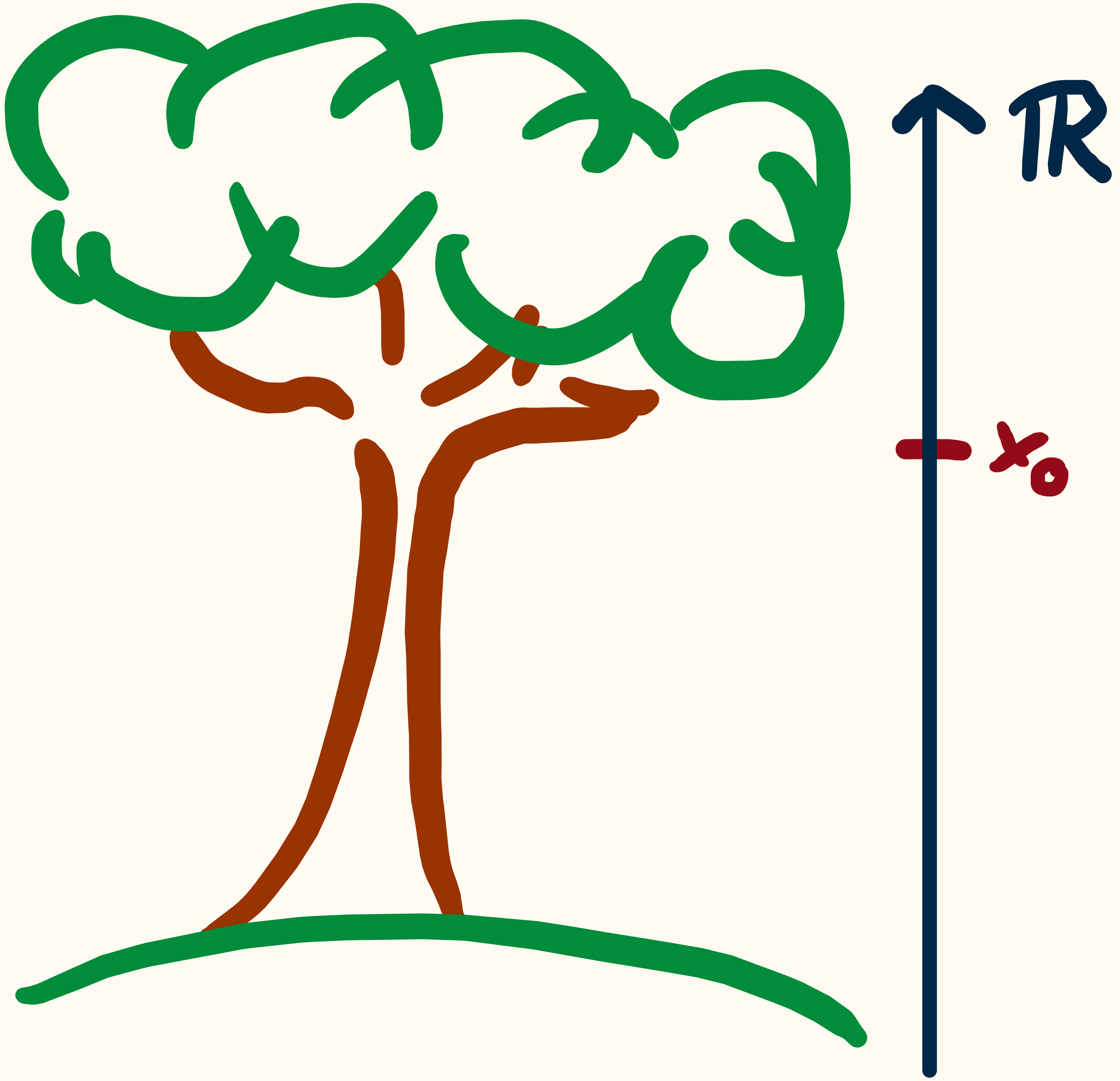


Classical Mechanics



$$\ddot{x} = -g$$

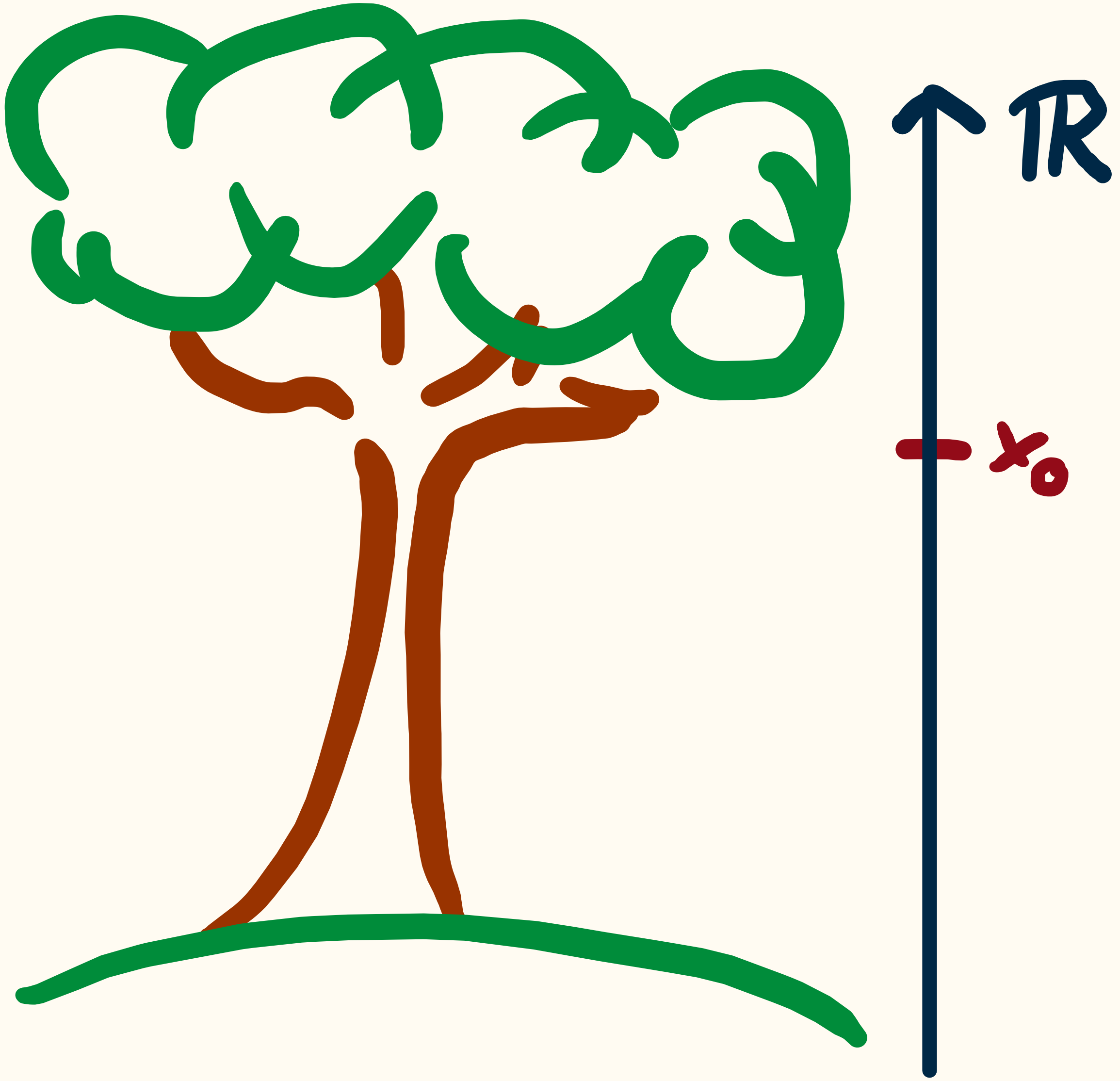
Classical Mechanics



$$\ddot{x} = -g$$

$$x(0) = x_0$$

Classical Mechanics

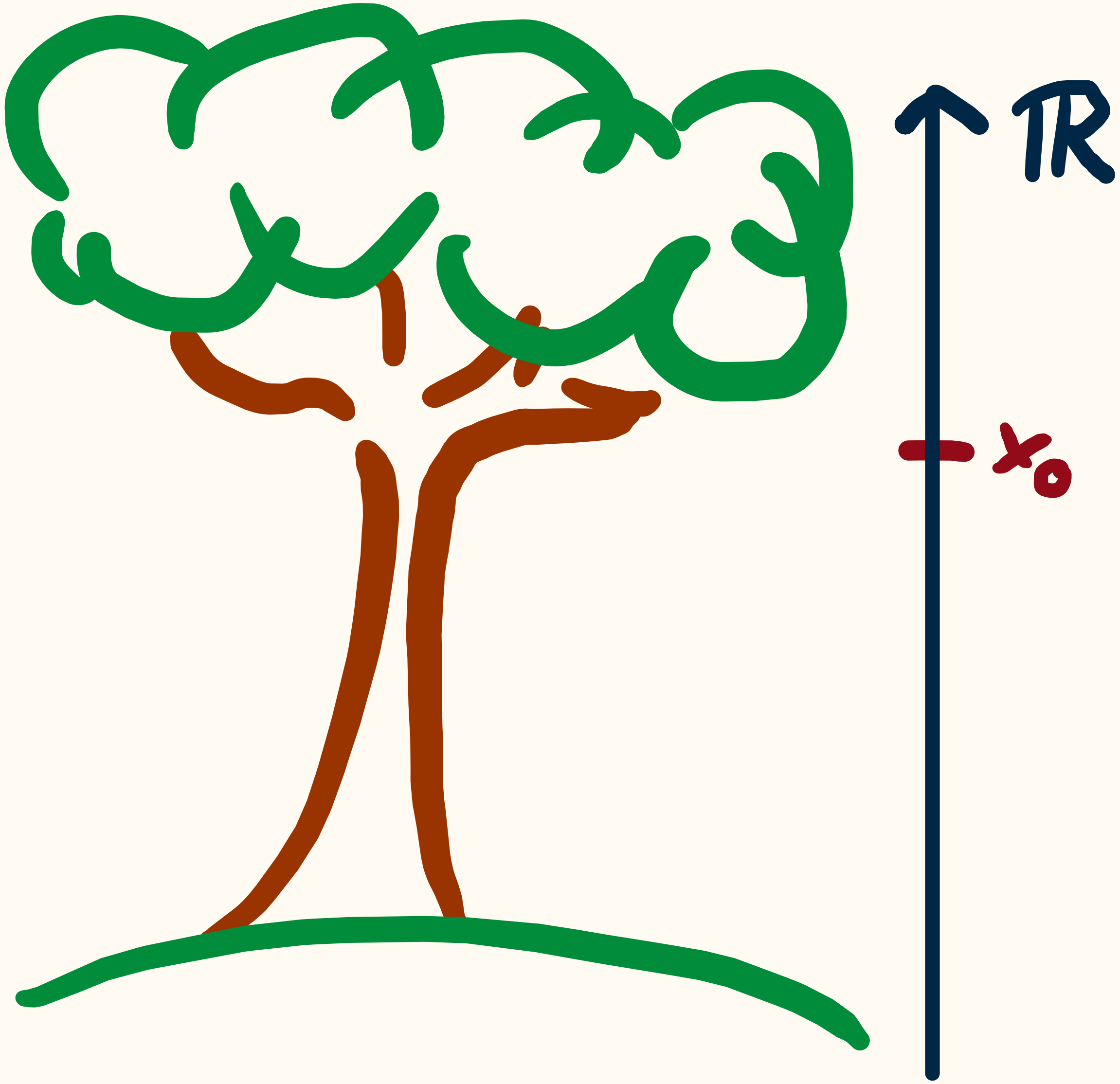


$$\ddot{x} = -g$$

$$x(0) = x_0$$

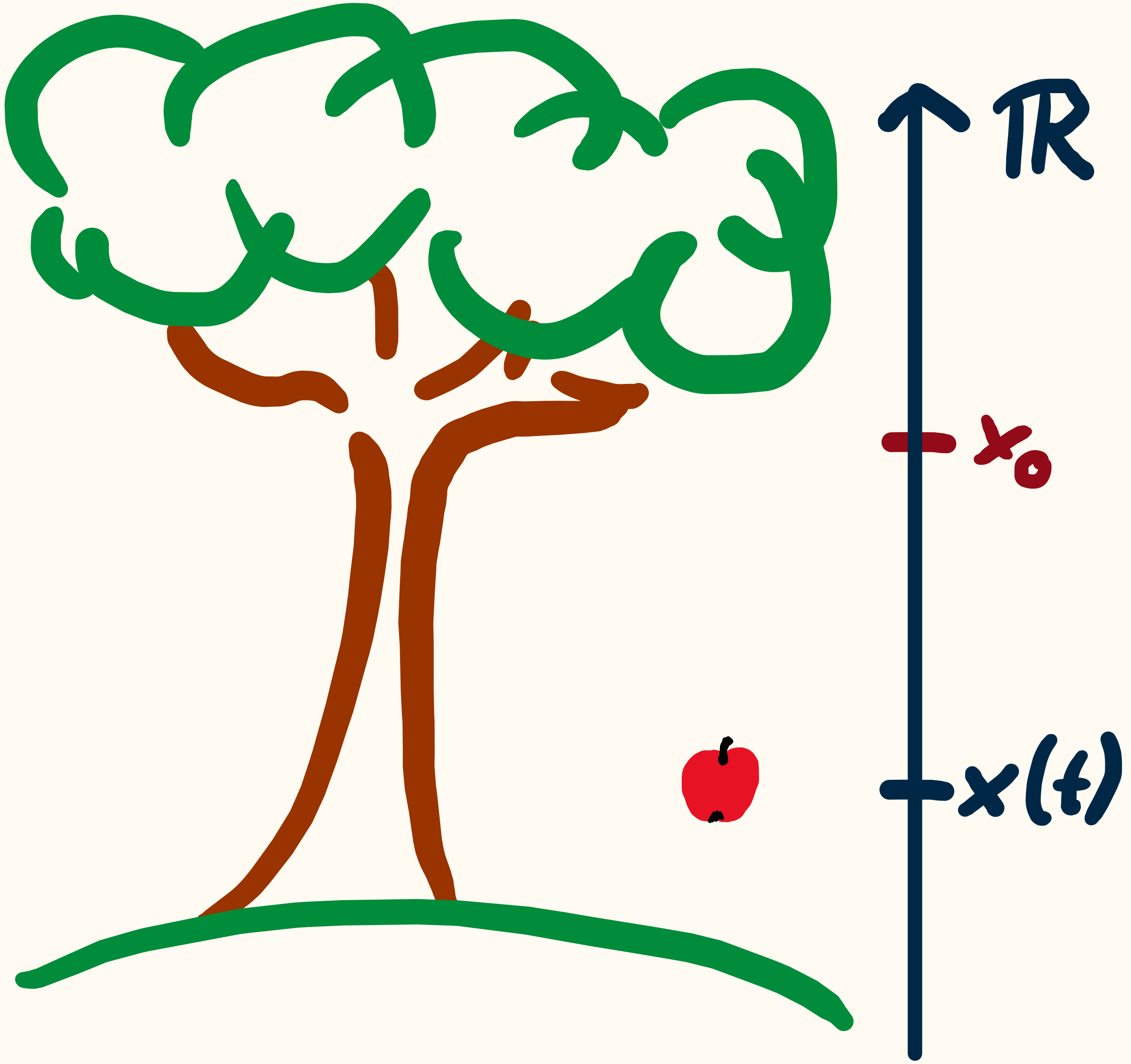
$$\dot{x}(0) = 0$$

Classical Mechanics



$$\left.\begin{aligned} \ddot{x} &= -g \\ x(0) &= x_0 \\ \dot{x}(0) &= 0 \end{aligned}\right\} \quad x(t) = x_0 - \frac{g}{2}t^2$$

Classical Mechanics



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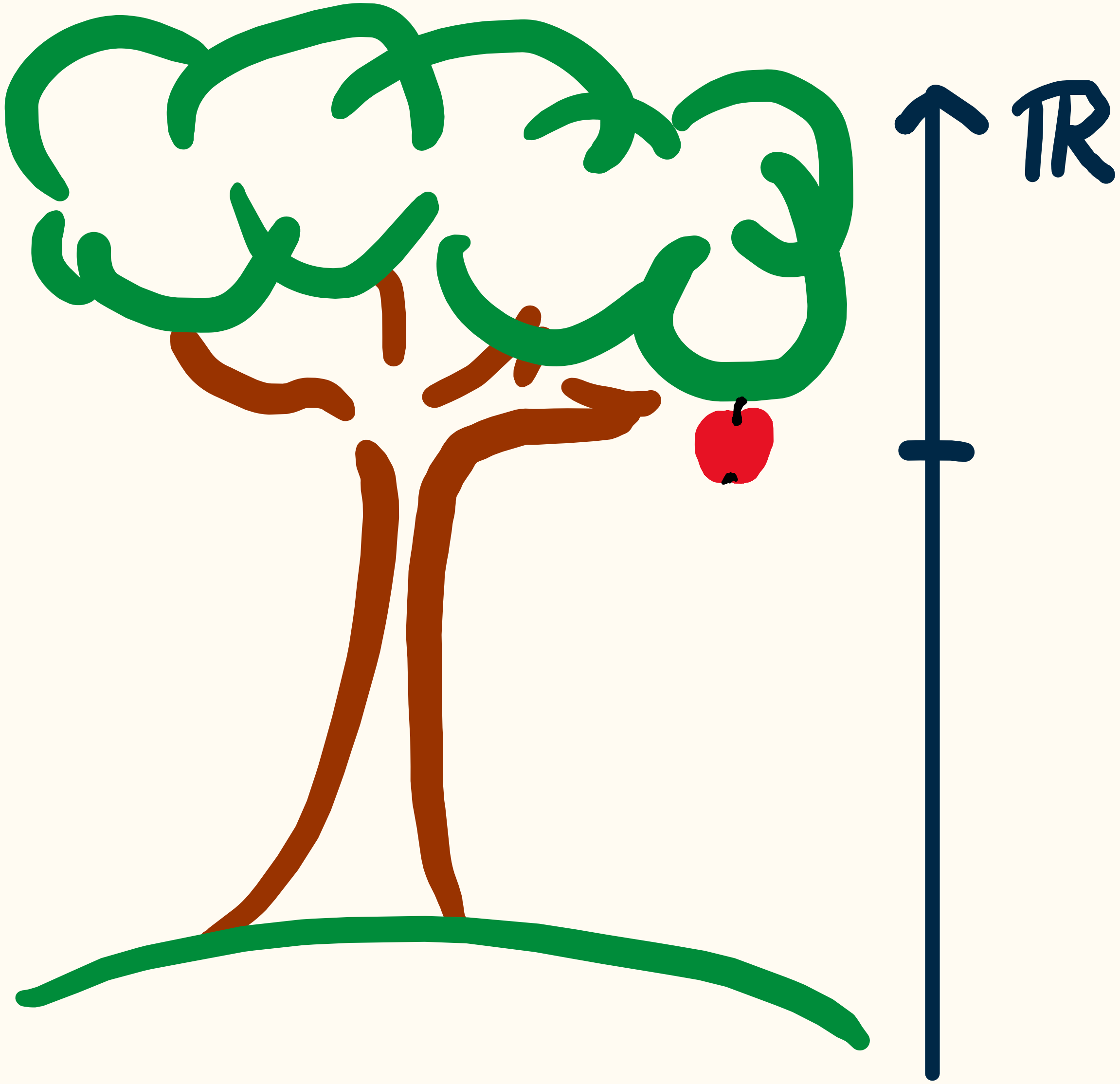
Quantum Mechanics



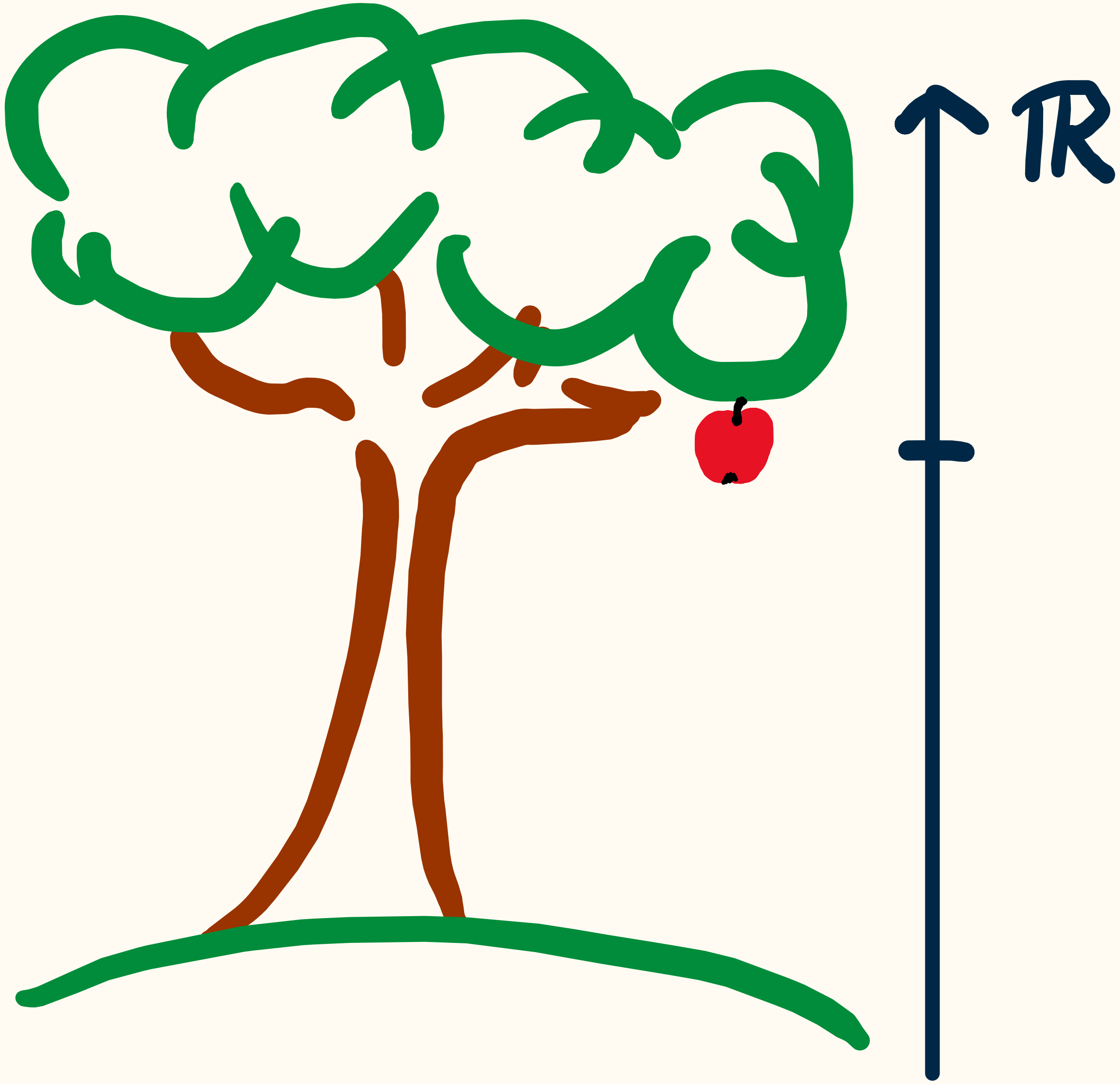
Slogan:

The apple takes every path

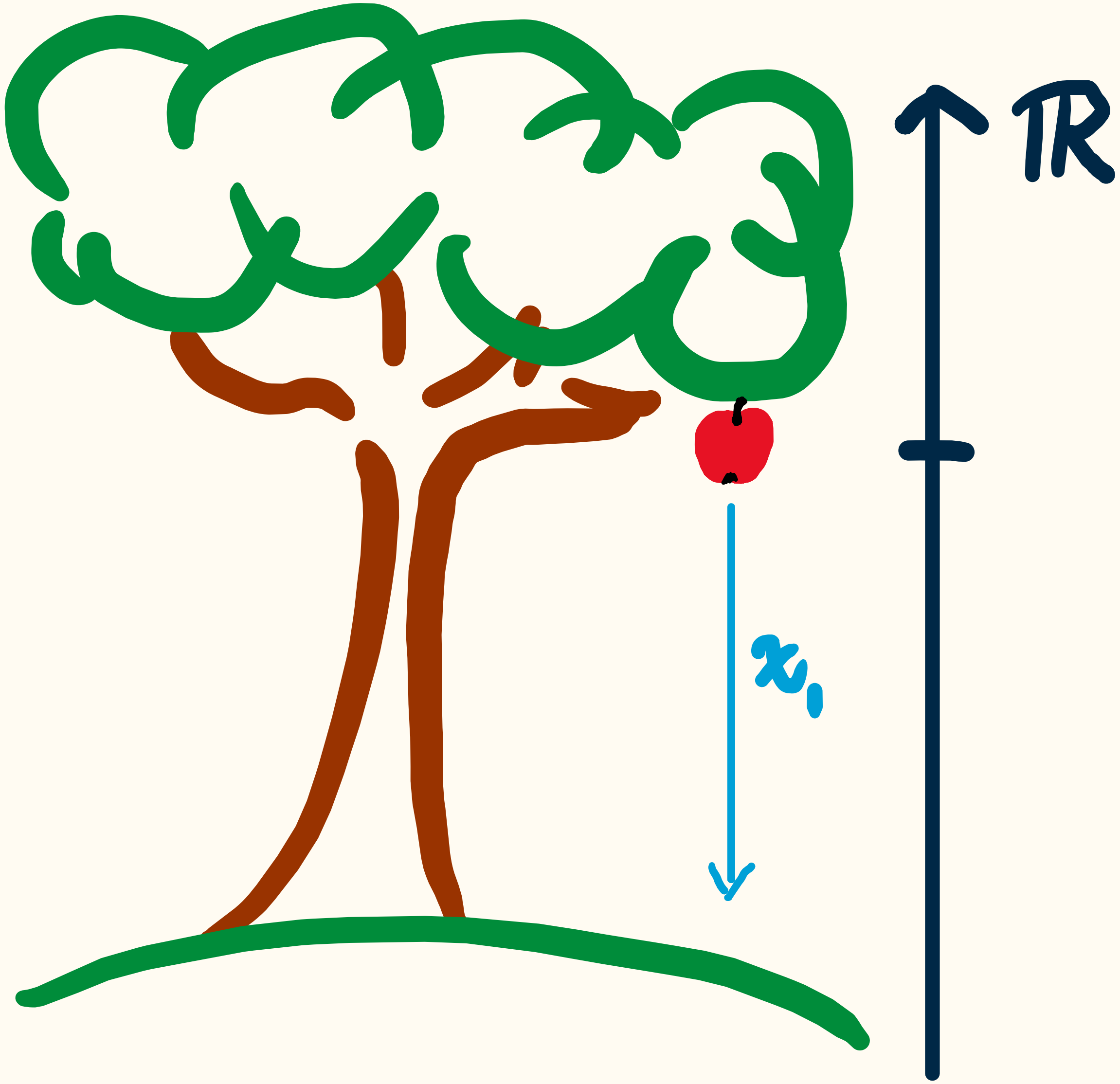
$$x : [0:t] \longrightarrow \mathbb{R}$$

with probability $P(x)$

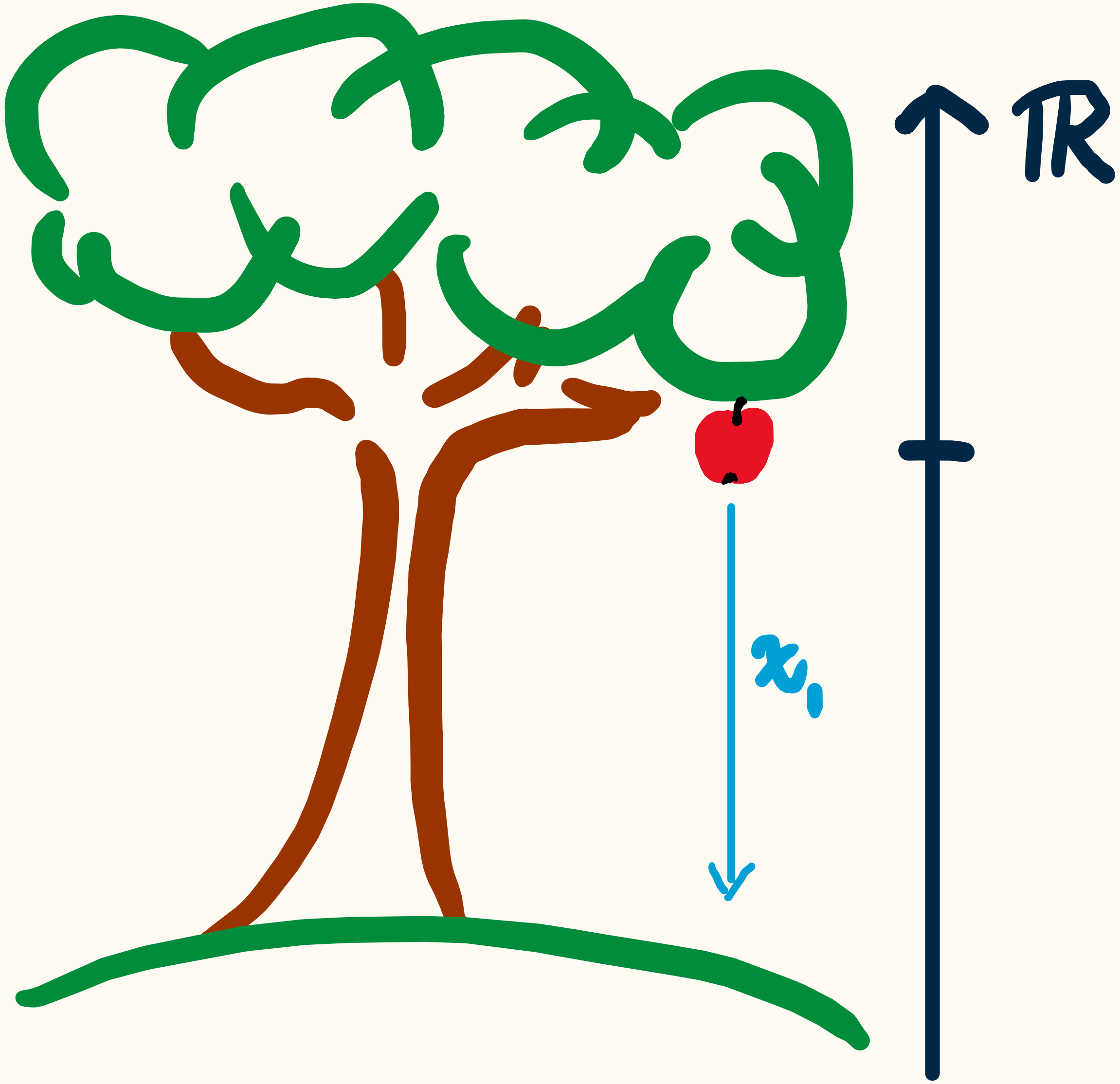
Quantum Mechanics

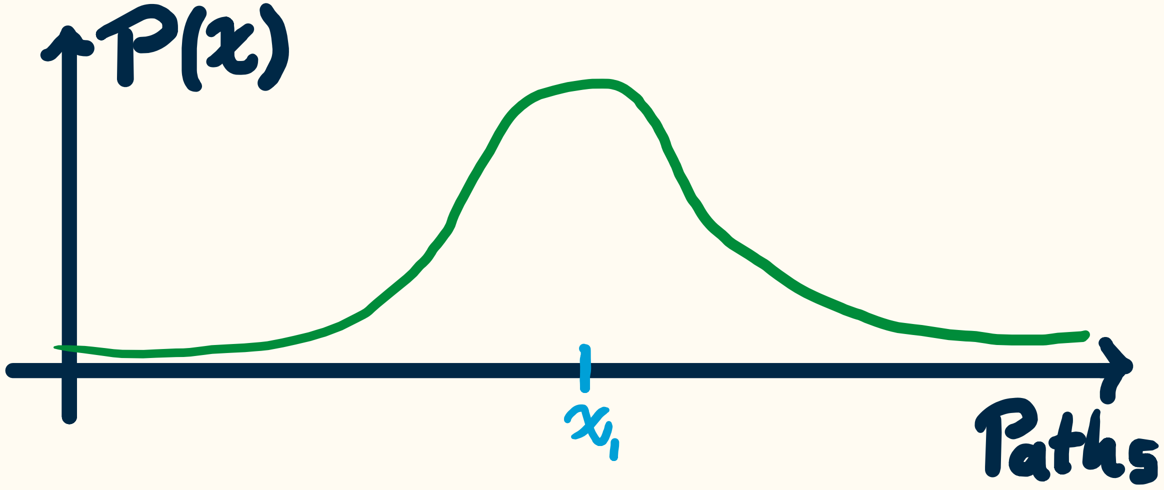


Quantum Mechanics

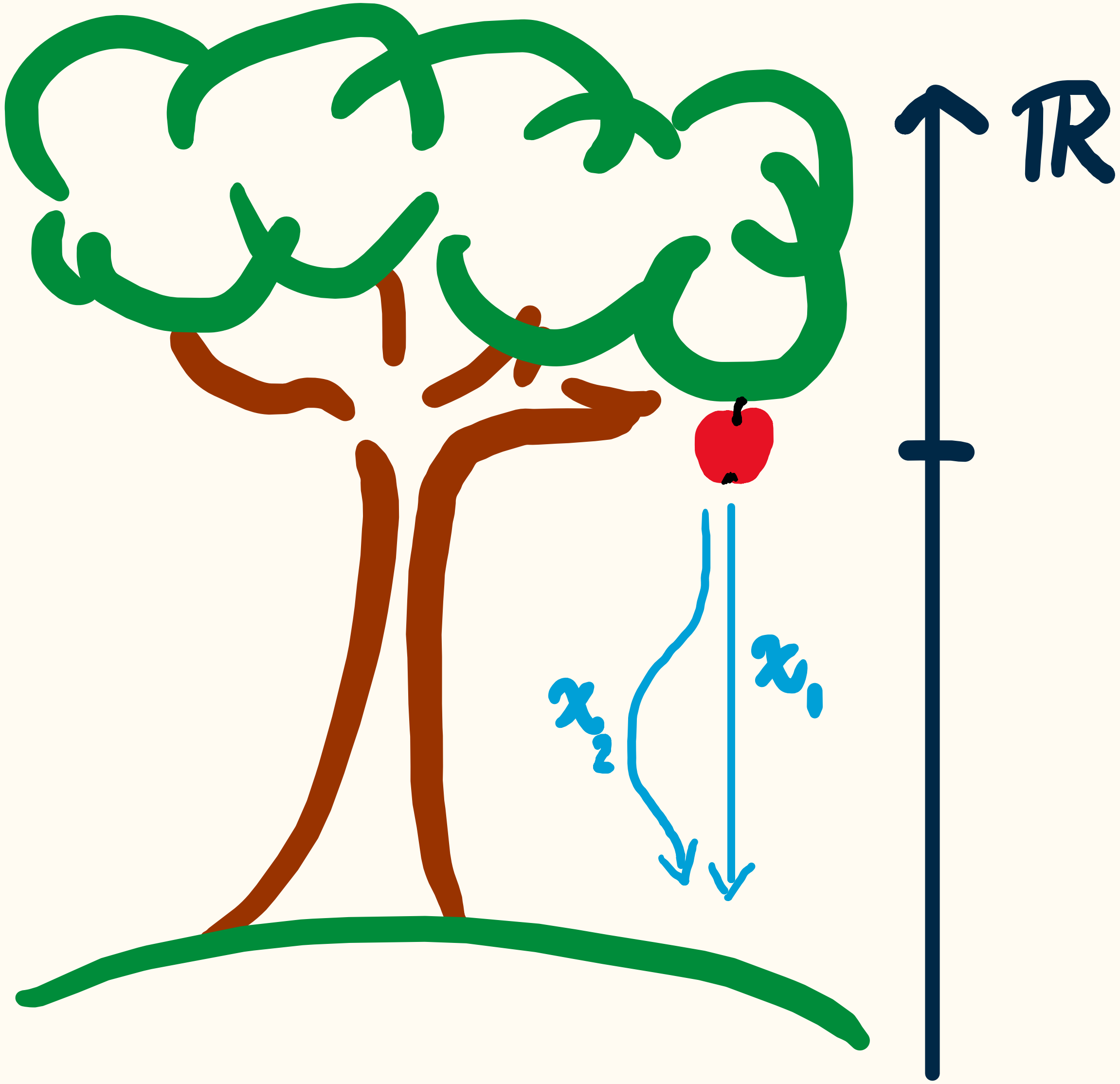


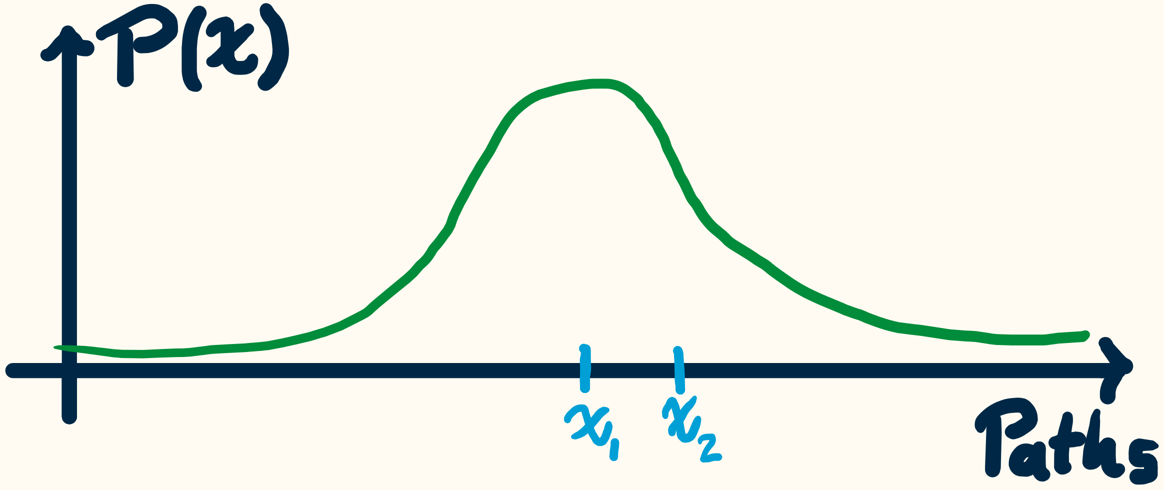
Quantum Mechanics



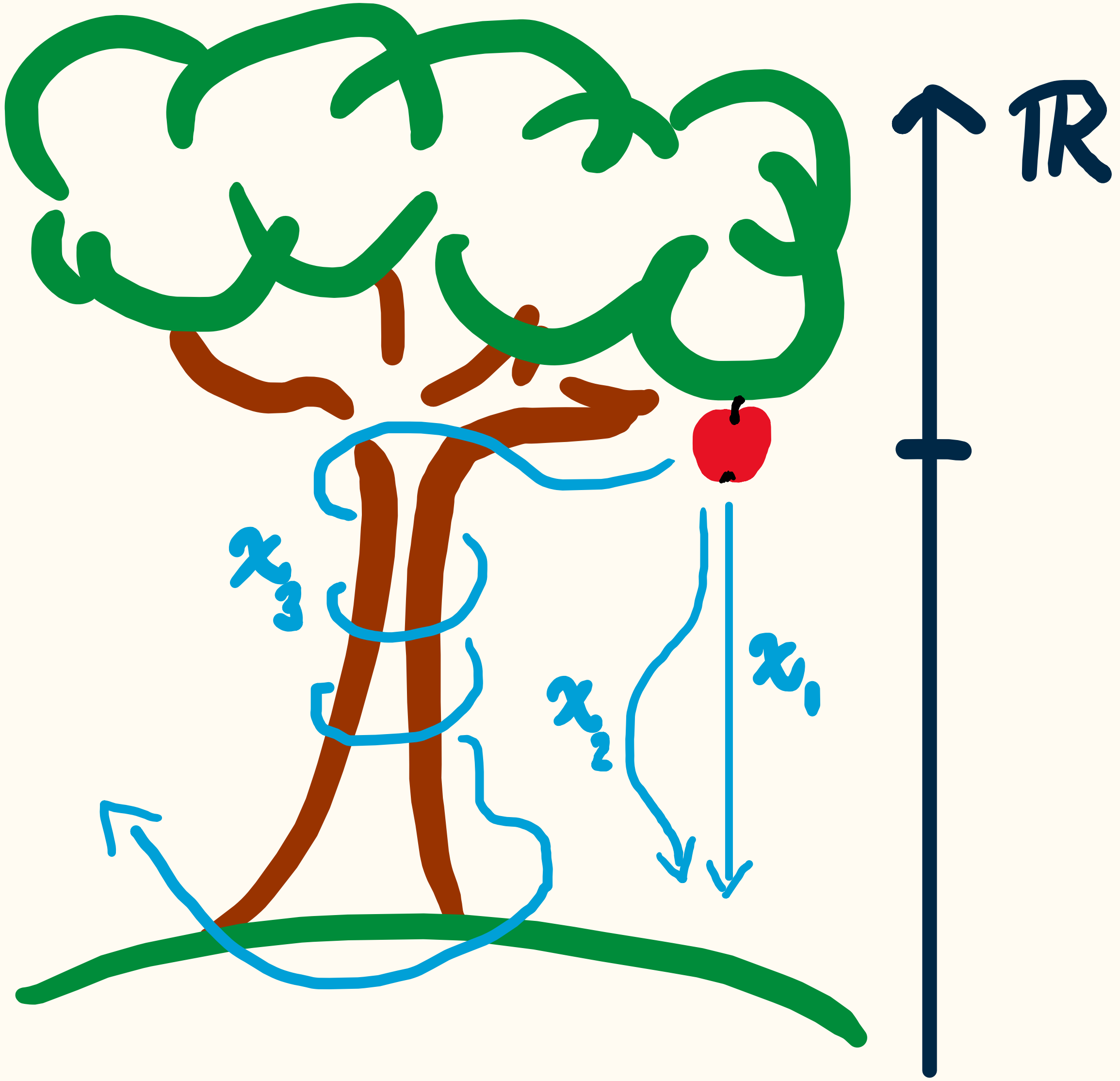


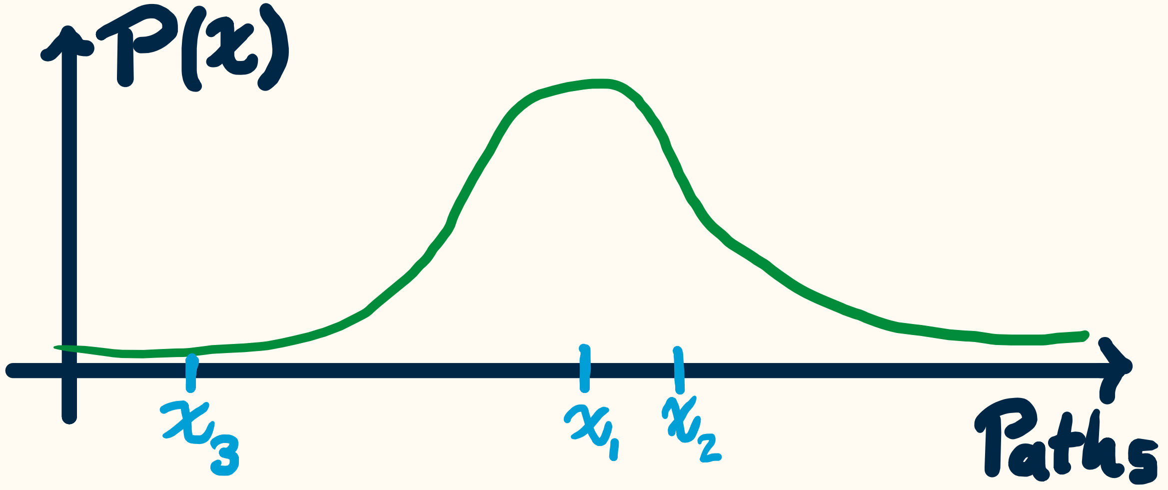
Quantum Mechanics





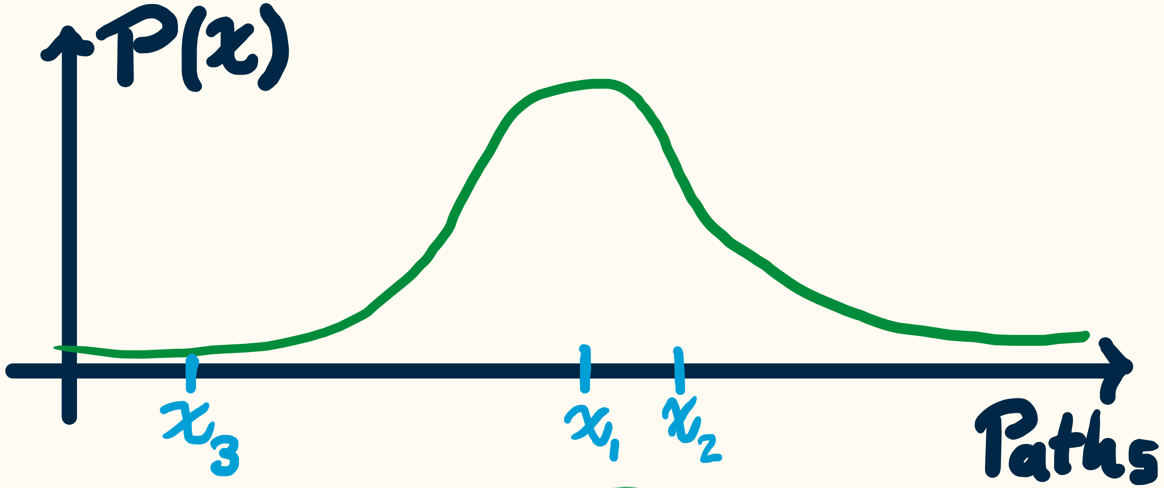
Quantum Mechanics

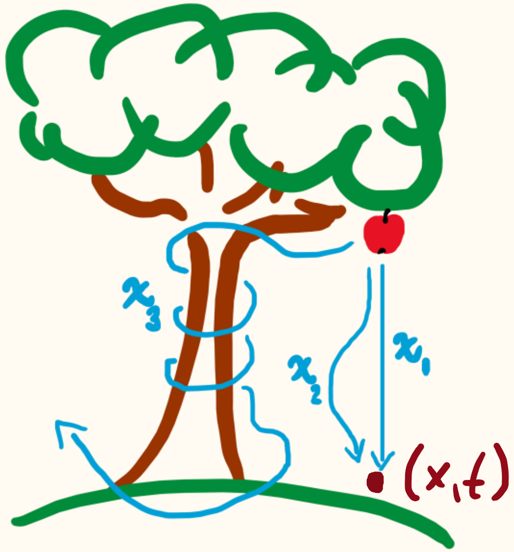




Quantum Mechanics

Probability to find apple at (x,t)

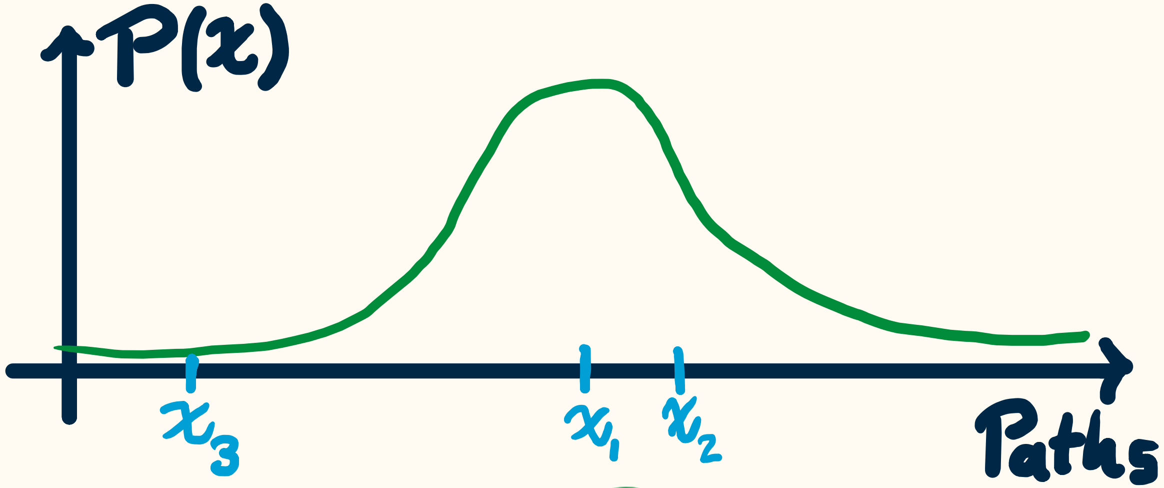


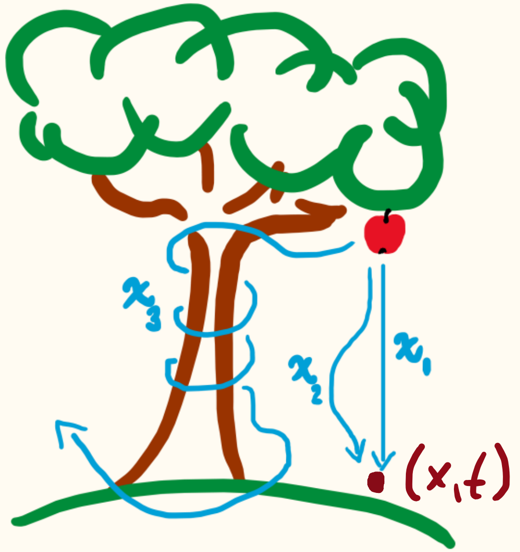


Quantum Mechanics

Probability to find apple at (x,t)

$$= \int_{\text{Paths} \cap \{x(t)=x\}} P(x)\, dx$$





Quantum Mechanics

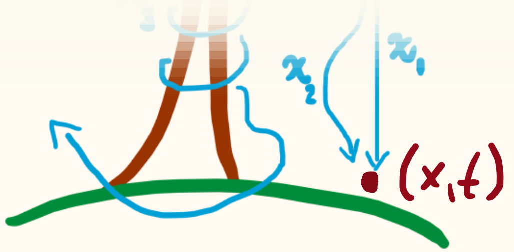
Probability to find apple at (x,t)

=

$$\int_{\text{Paths} \,\cap\, \{x(t)=x\}} P(x)\, dx$$

Problem!

$\nexists$ measure "dx" on the 'space of Paths'



Quantum Mechanics

Probability to find apple at (x,t)

$$=$$

$$\int_{\text{Paths} \cap \{x(t)=x\}} P(x)\, \mathcal{D}x$$

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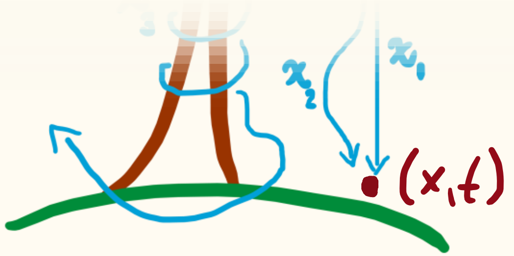
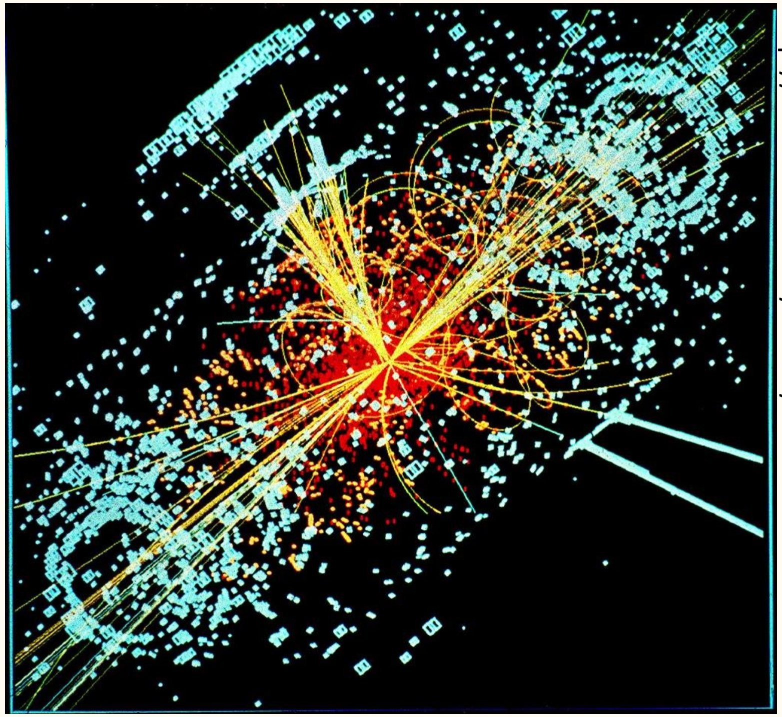


Quantum Field Theory



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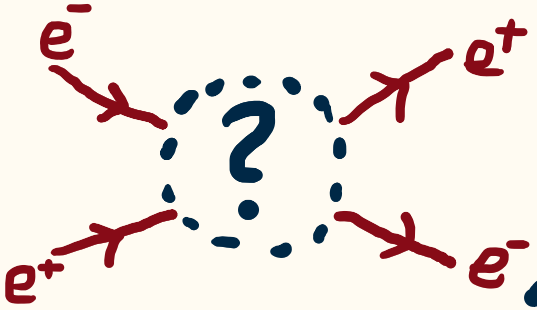
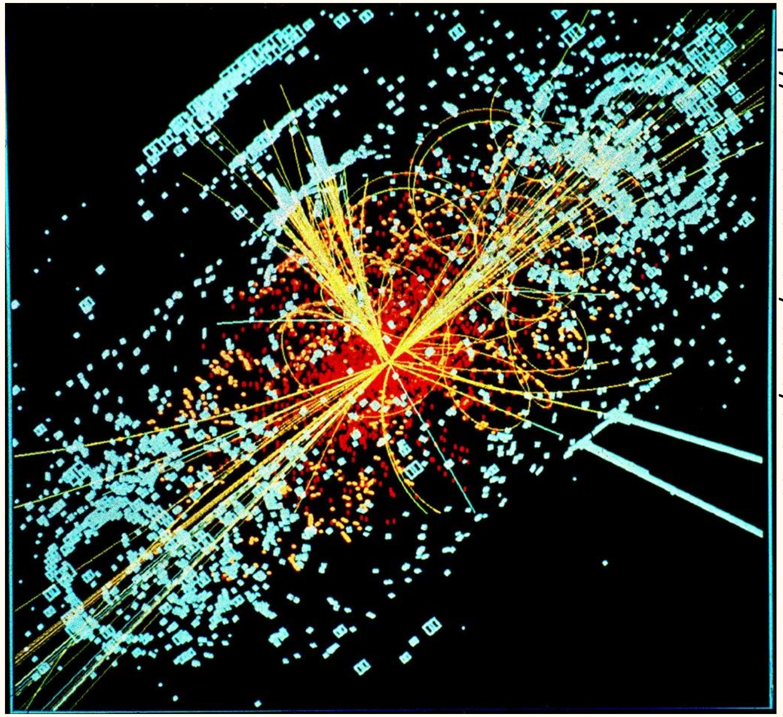
Quantum Field Theory

Probability()



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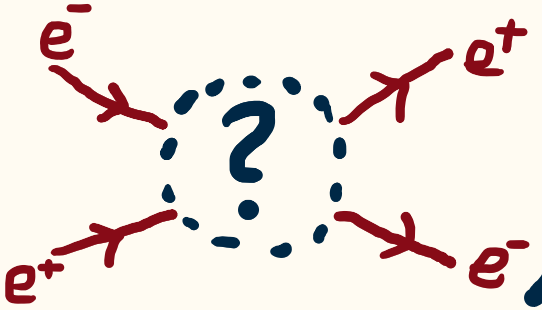
Quantum Field Theory

Probability(


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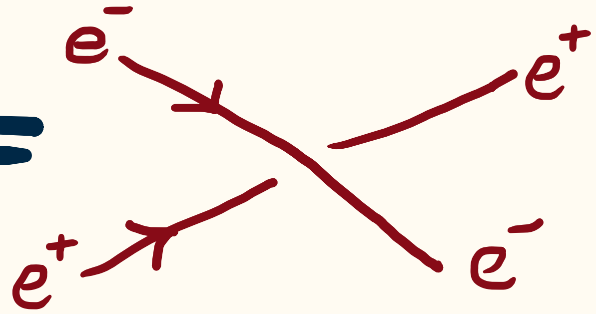

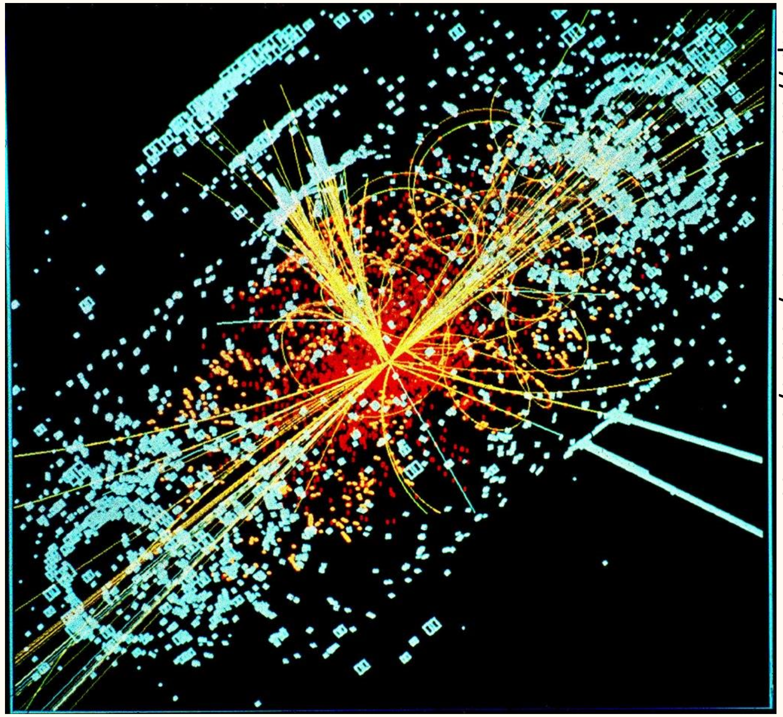

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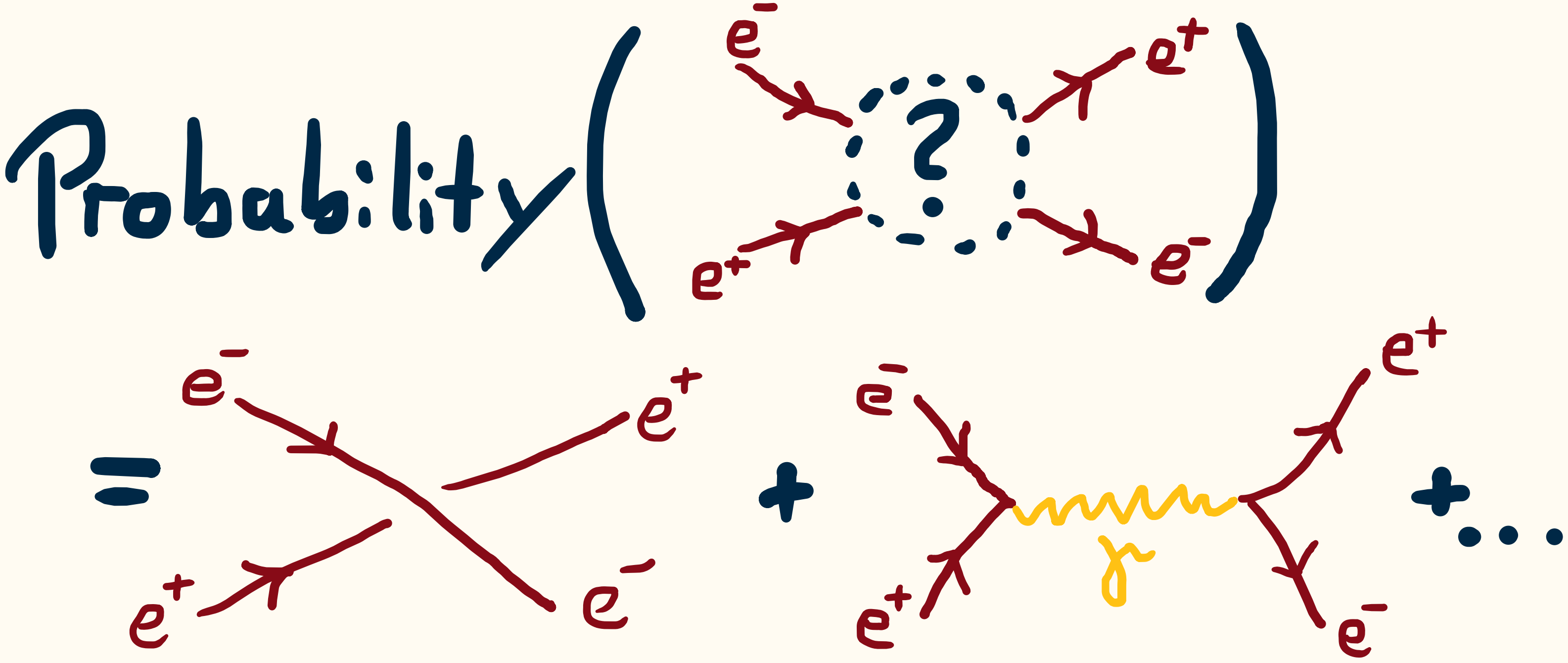
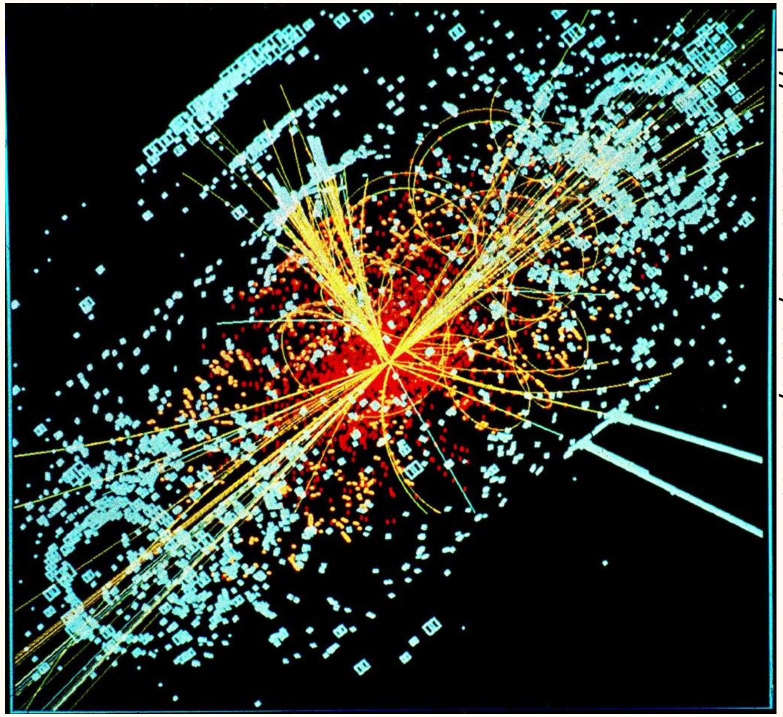
Quantum Field Theory





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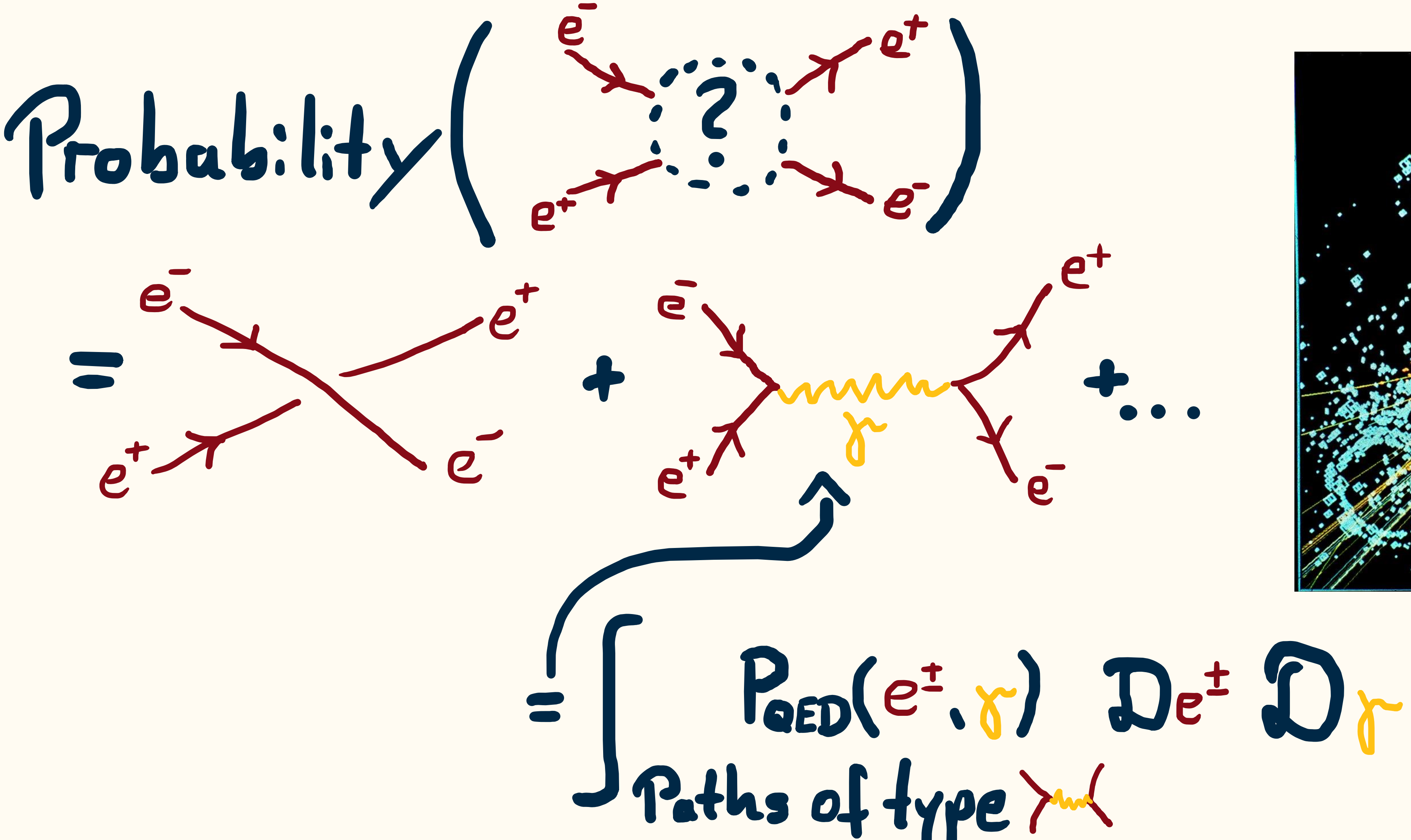
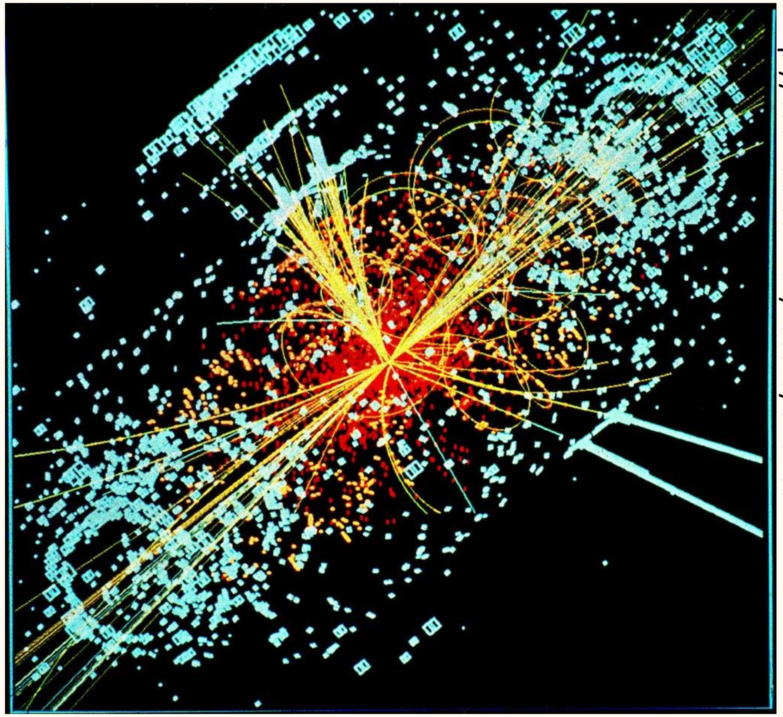
Quantum Field Theory





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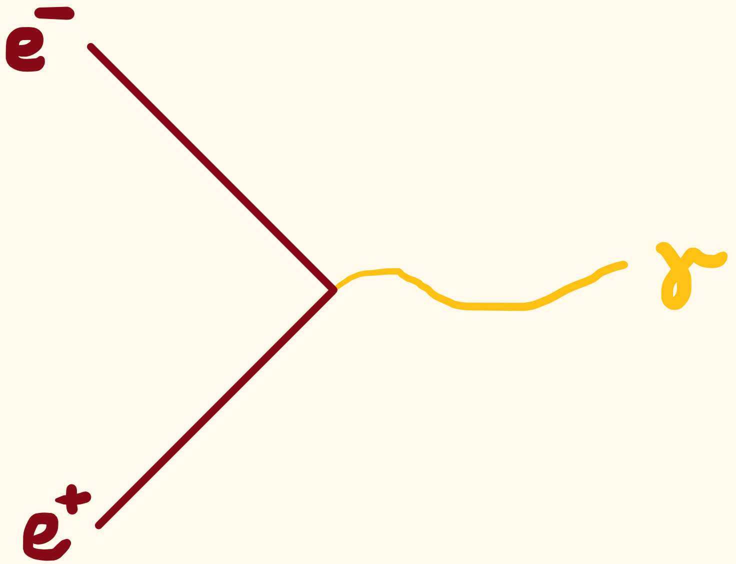
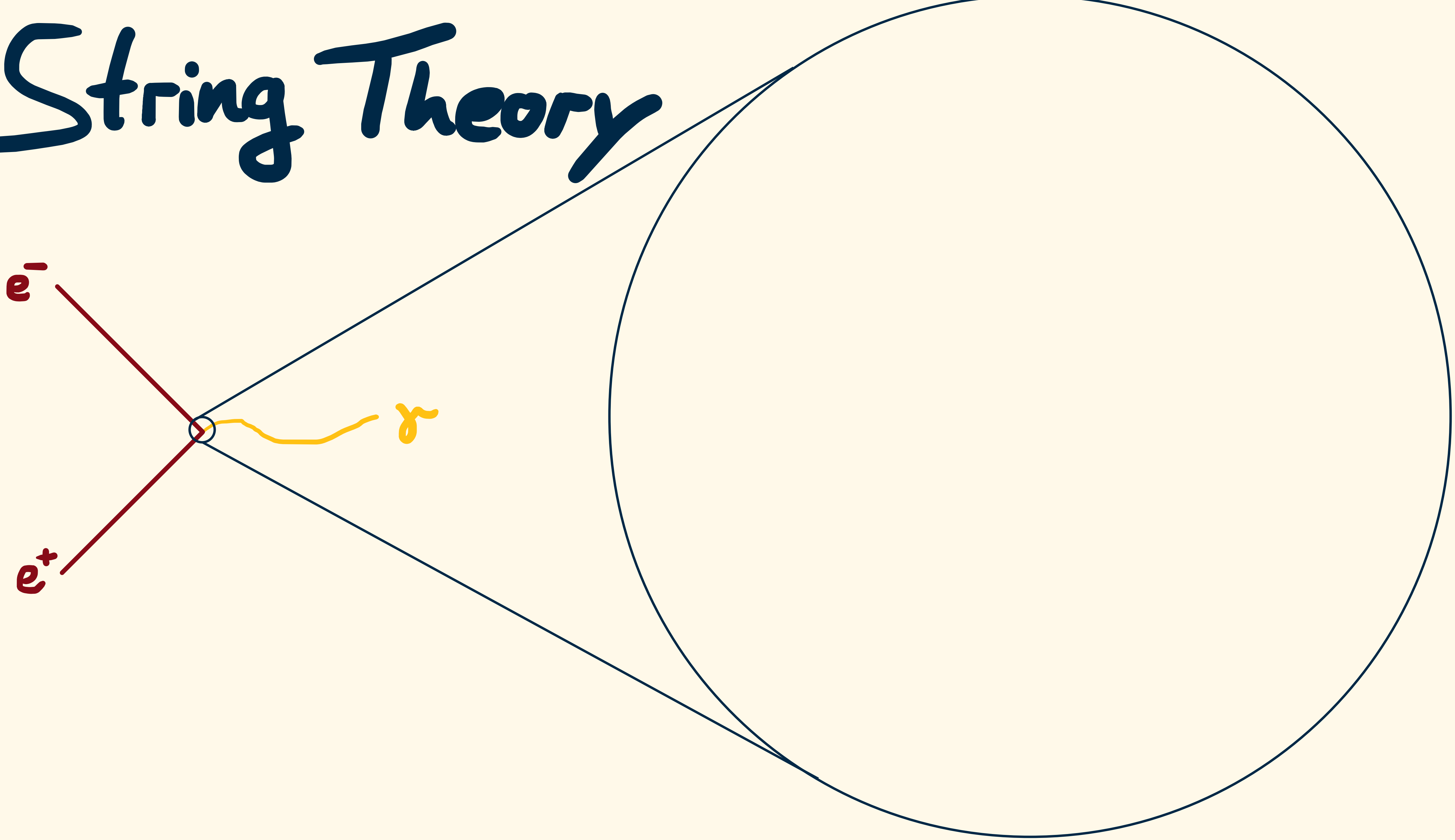
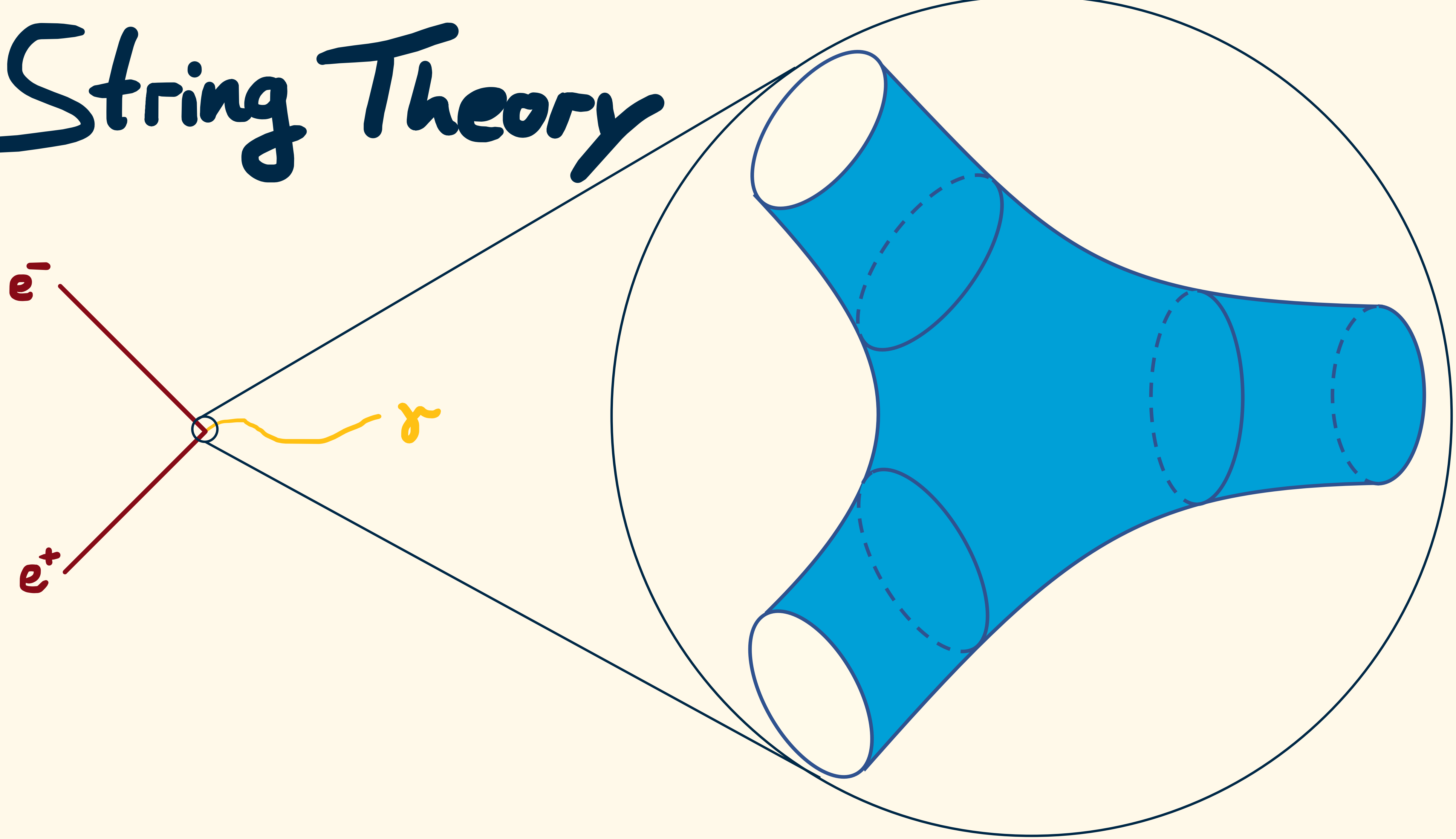
String Theory



String Theory



String Theory

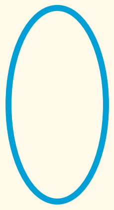


Point

String

Point

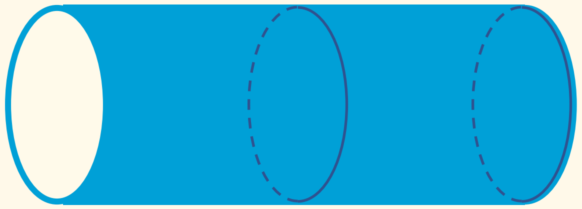
String



Point

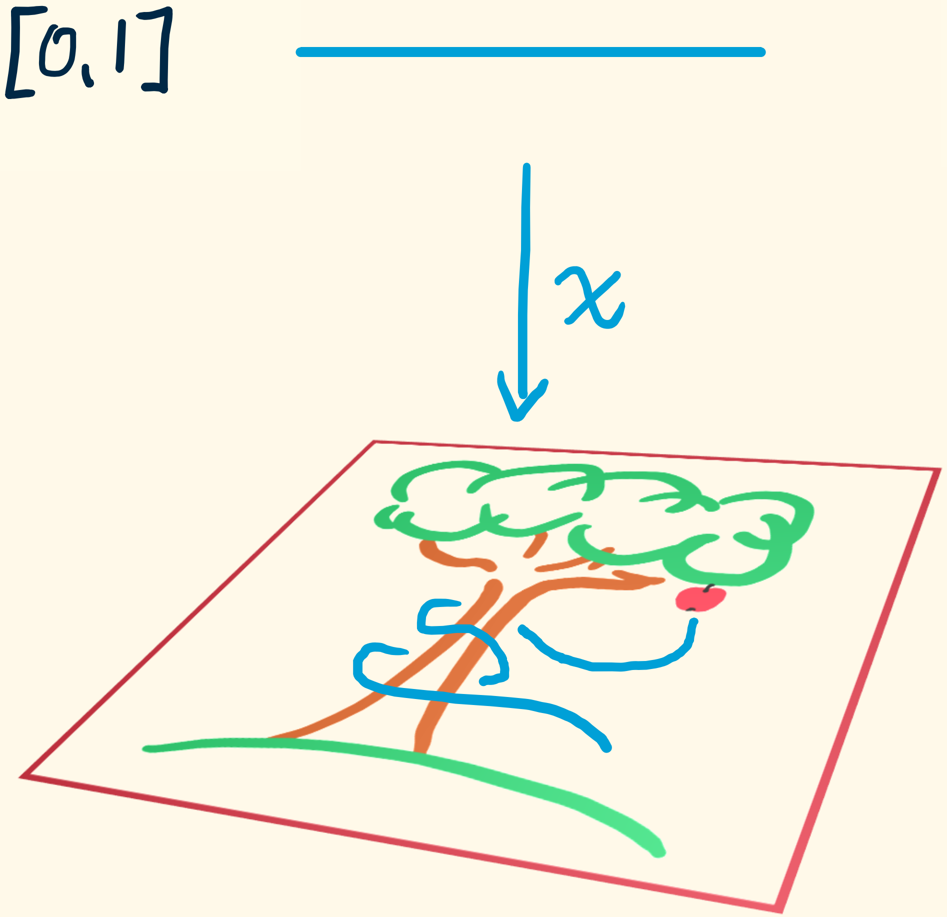
$[0,1]$

String



$S^1 \times [0,1]$

Point



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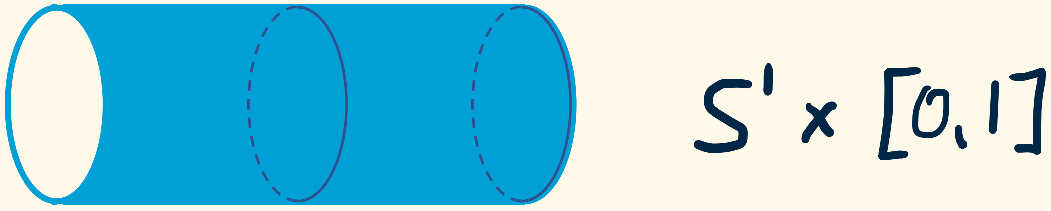
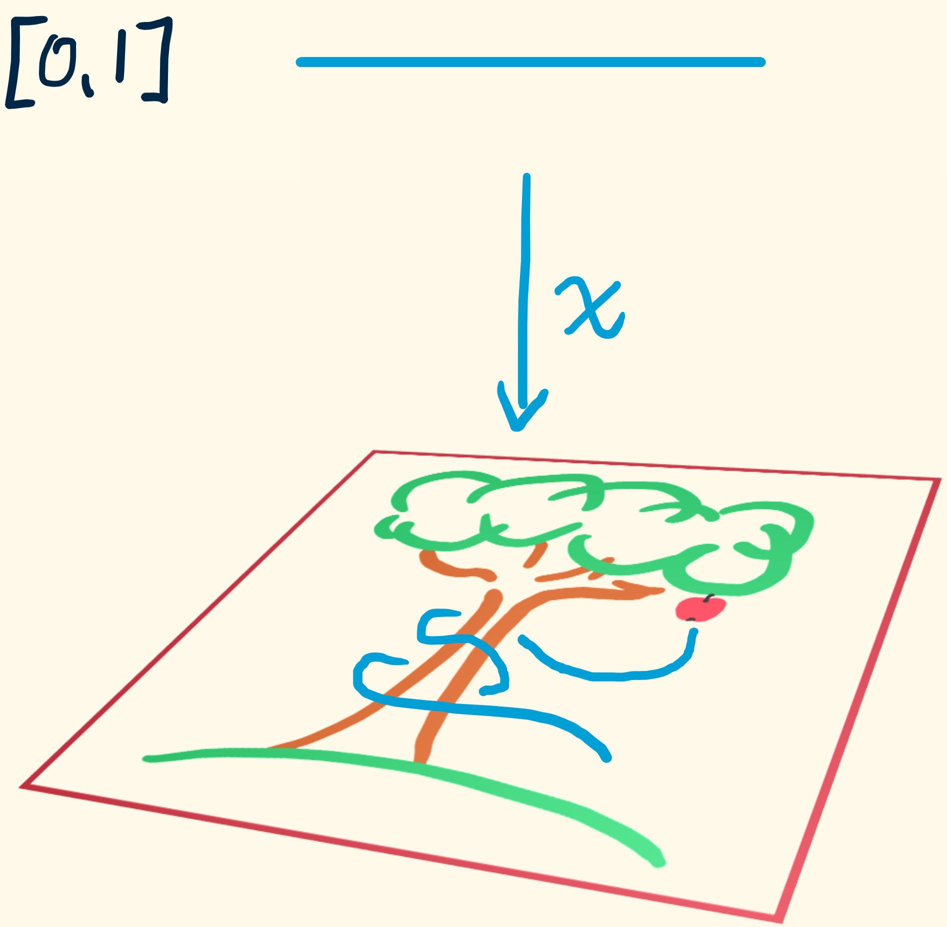


Point



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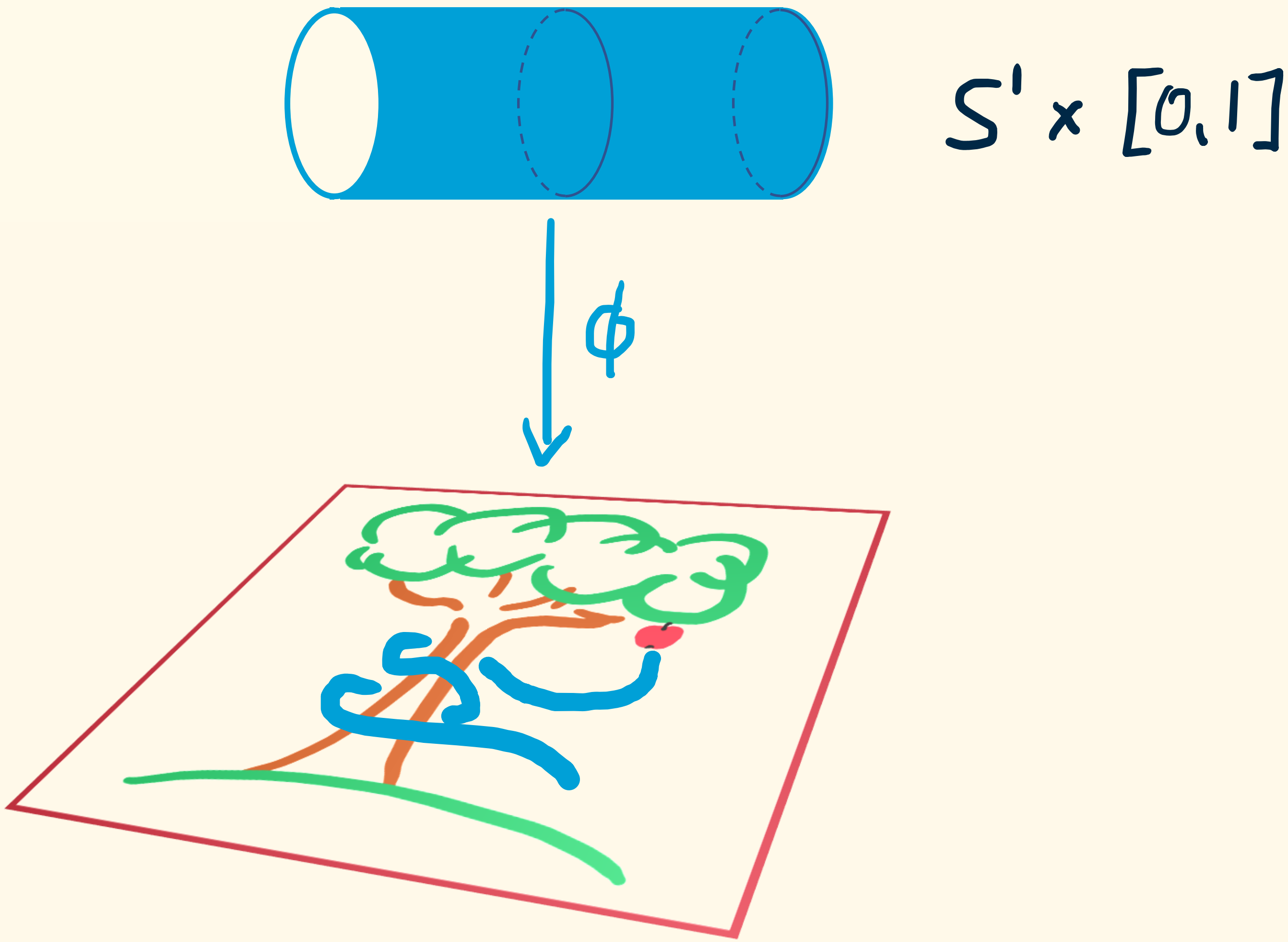

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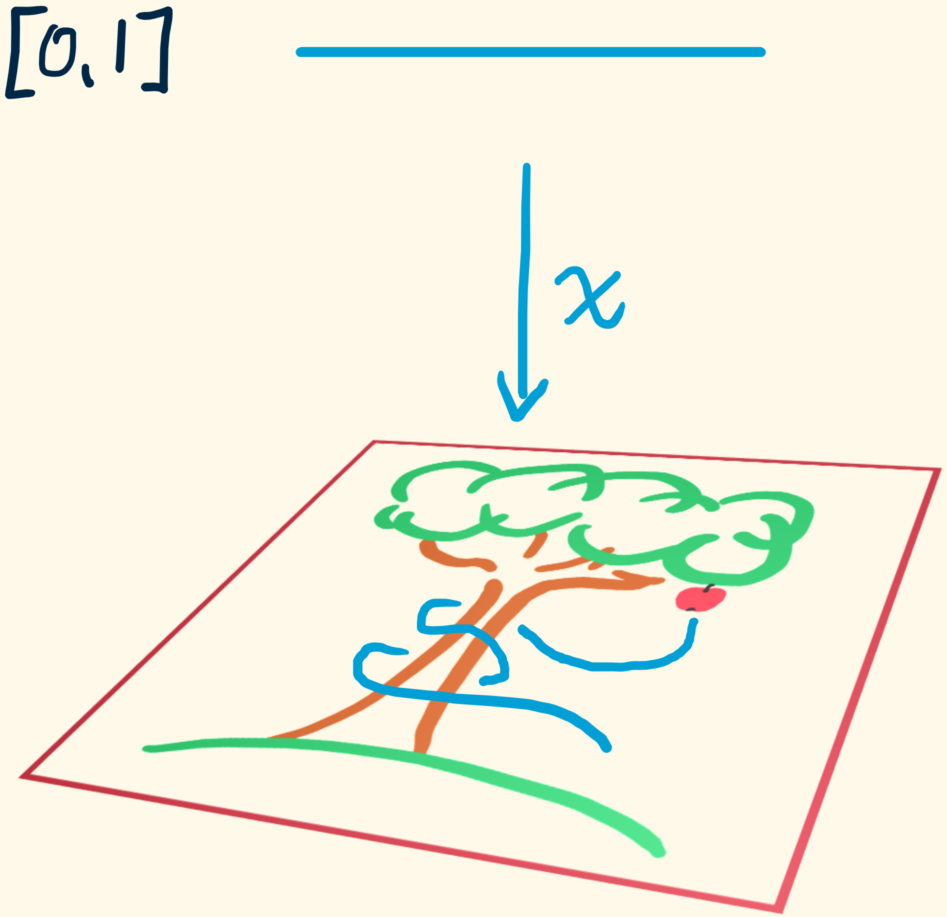
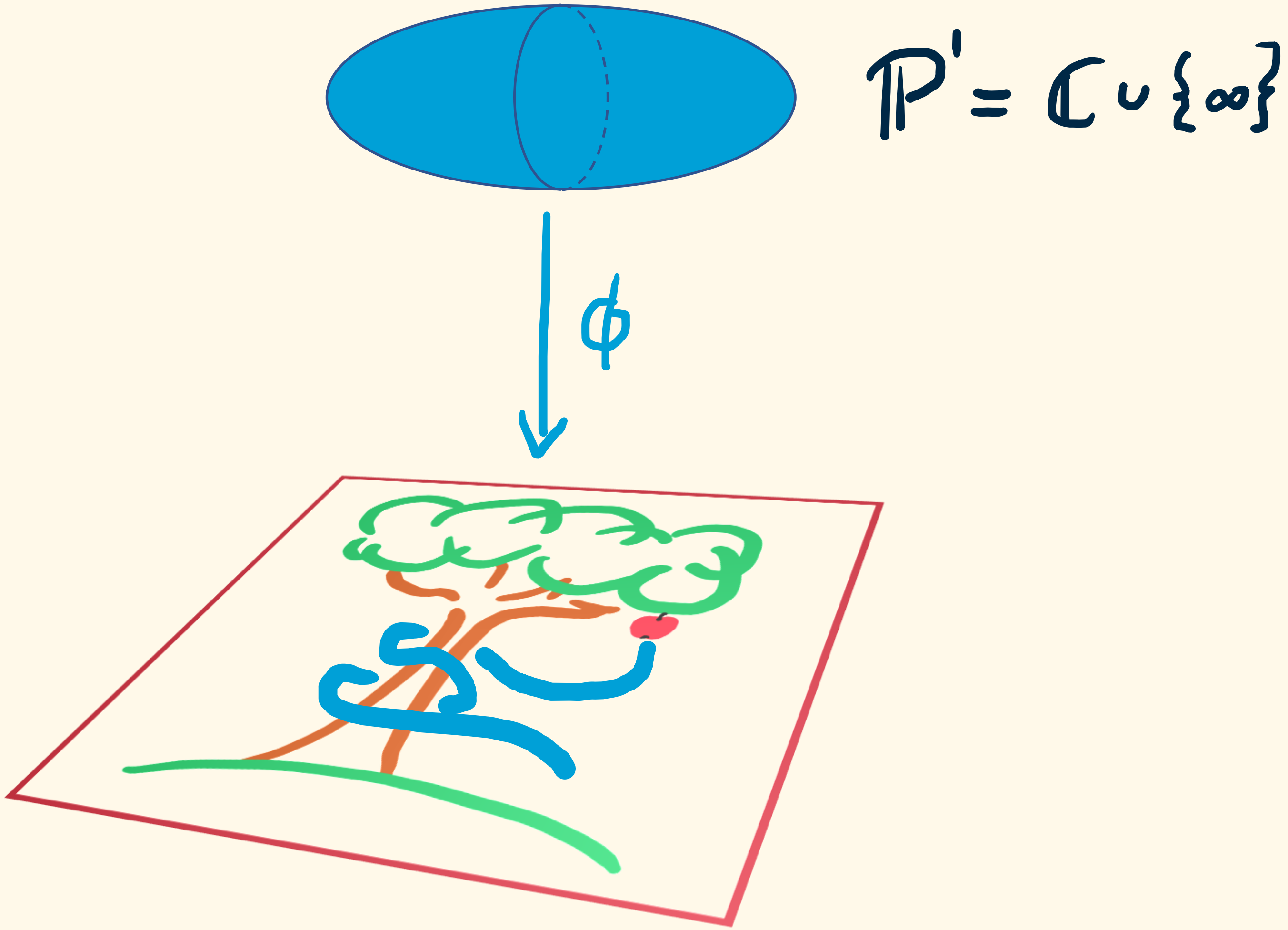


String



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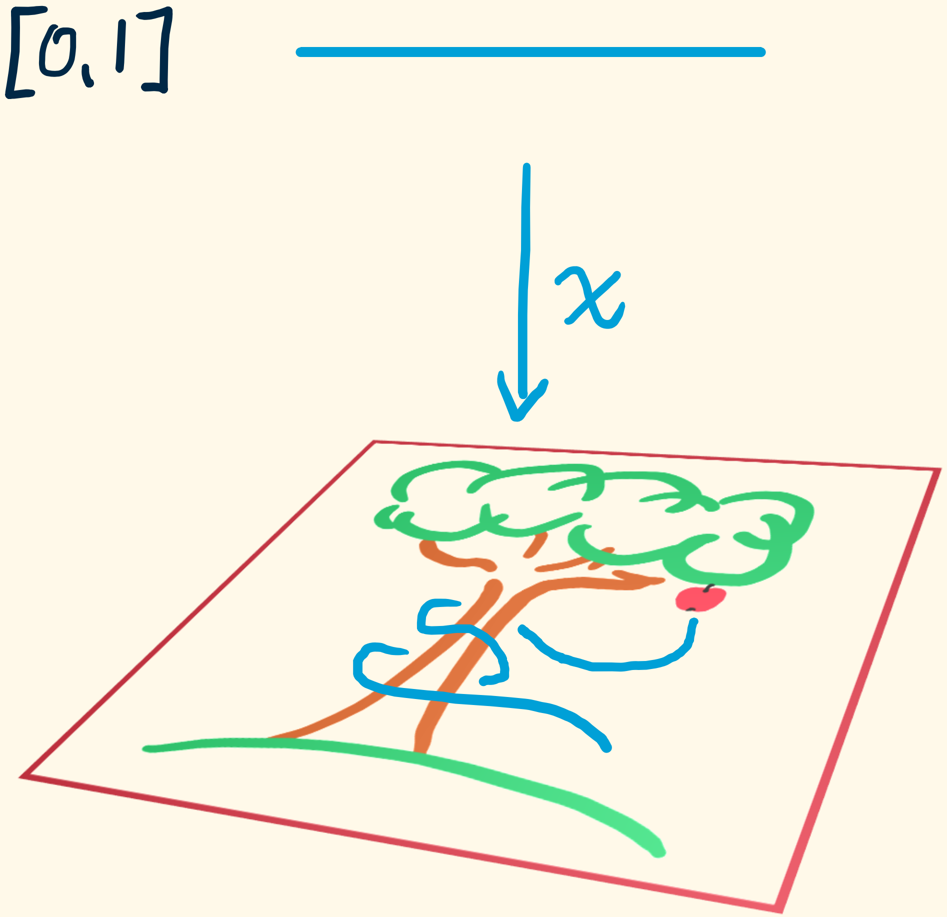

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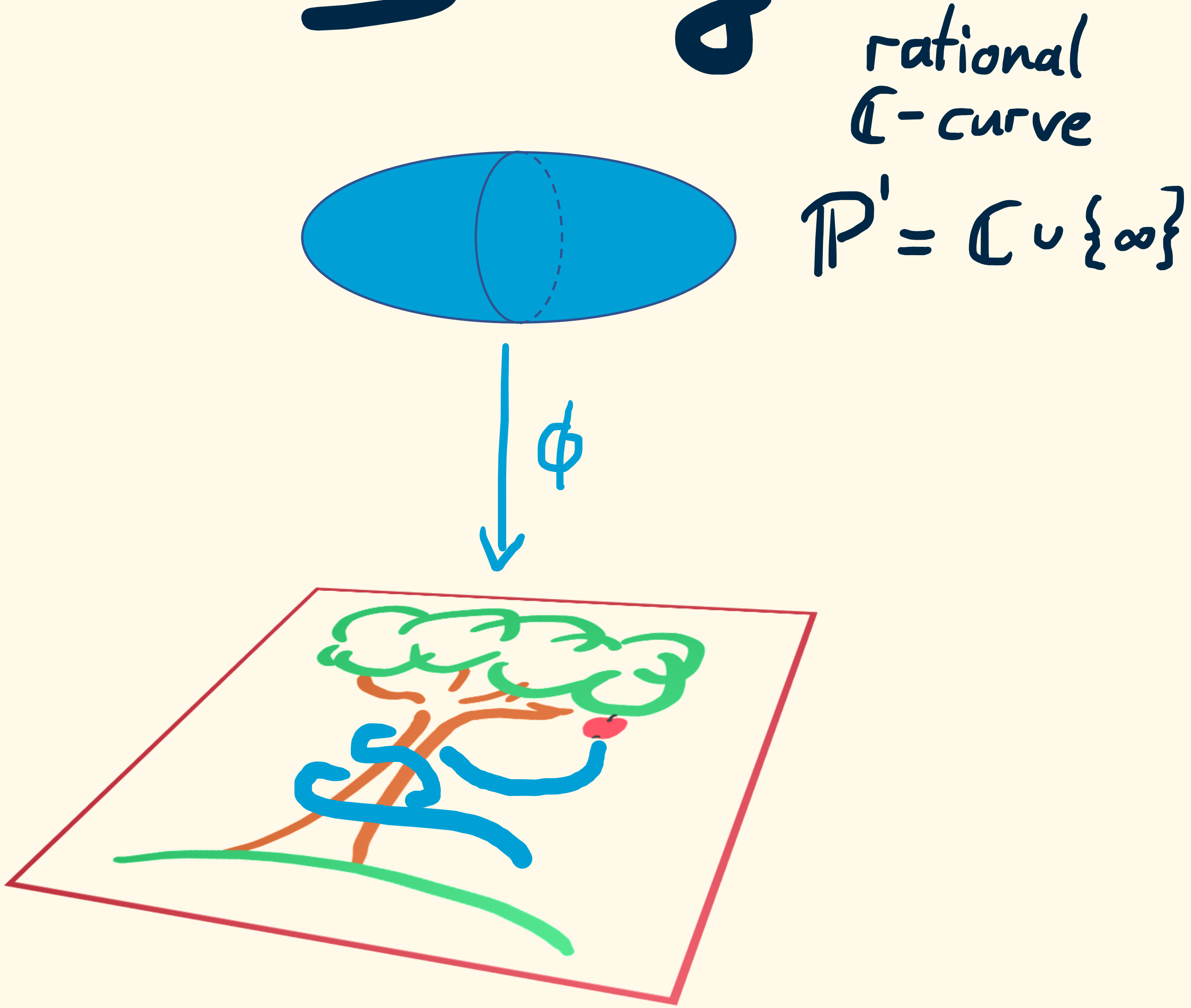

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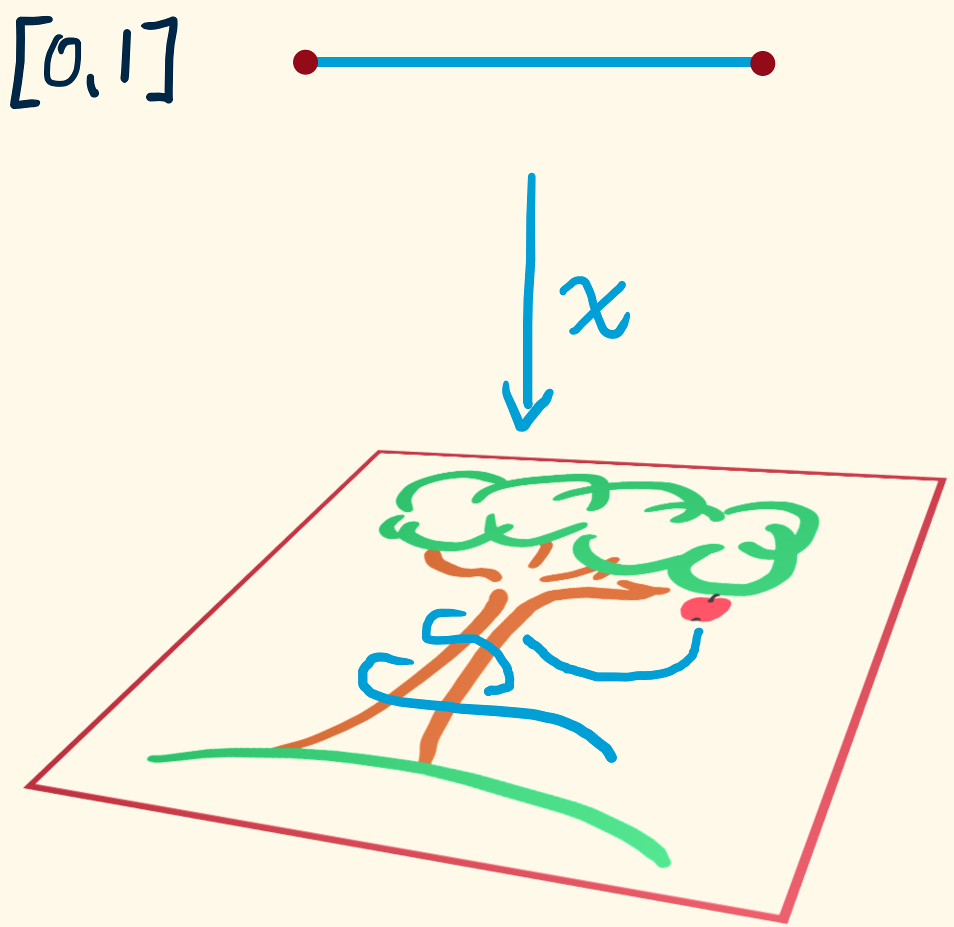

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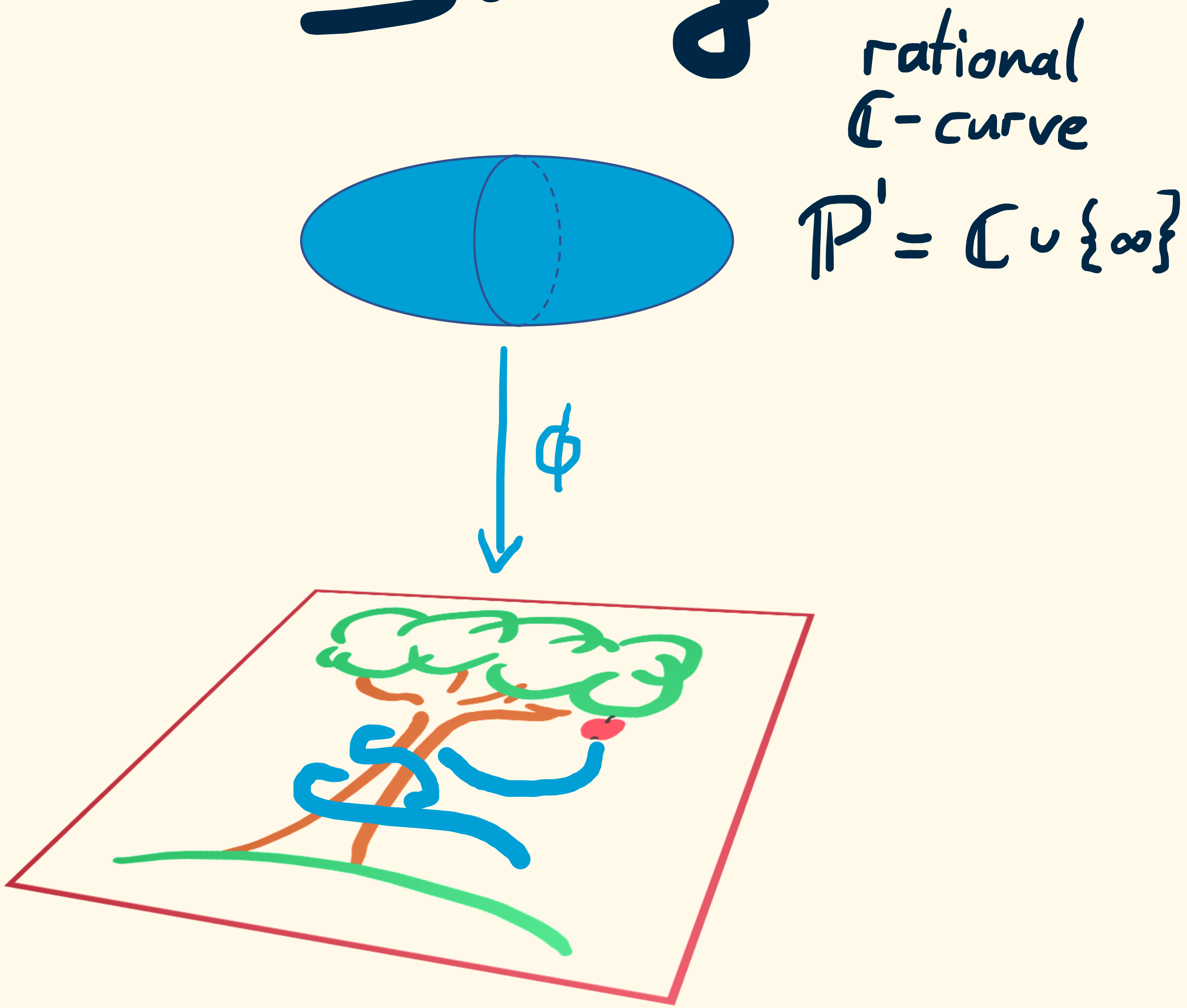

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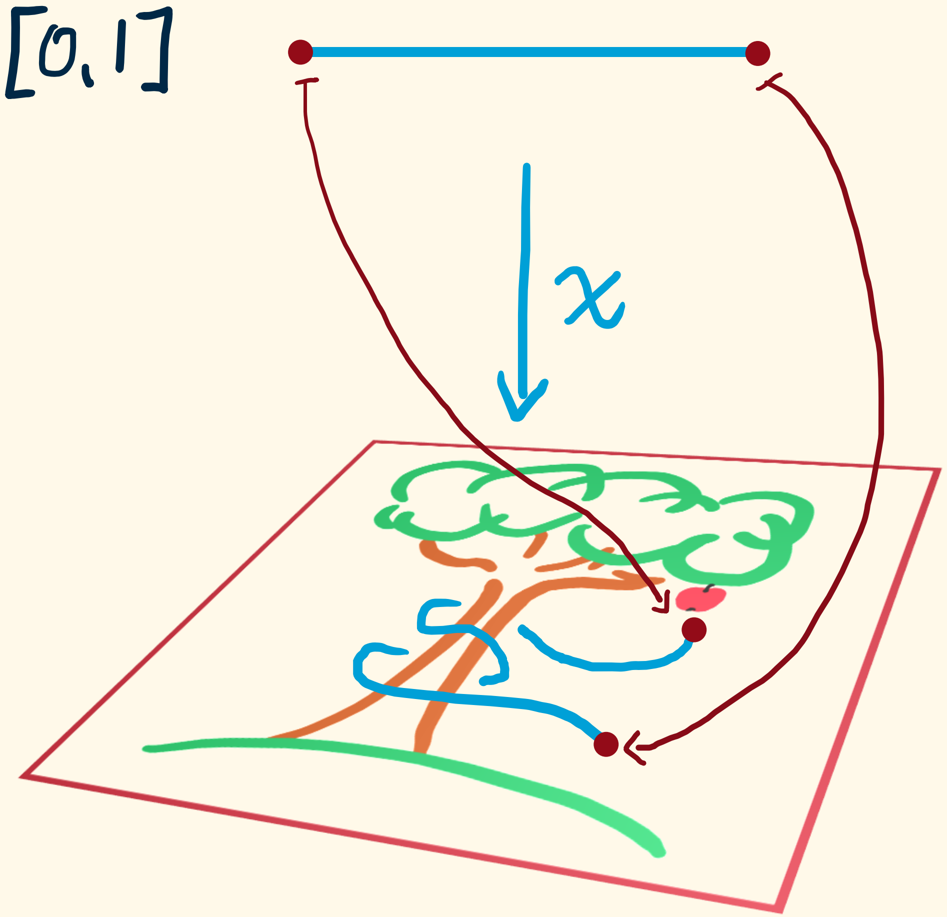

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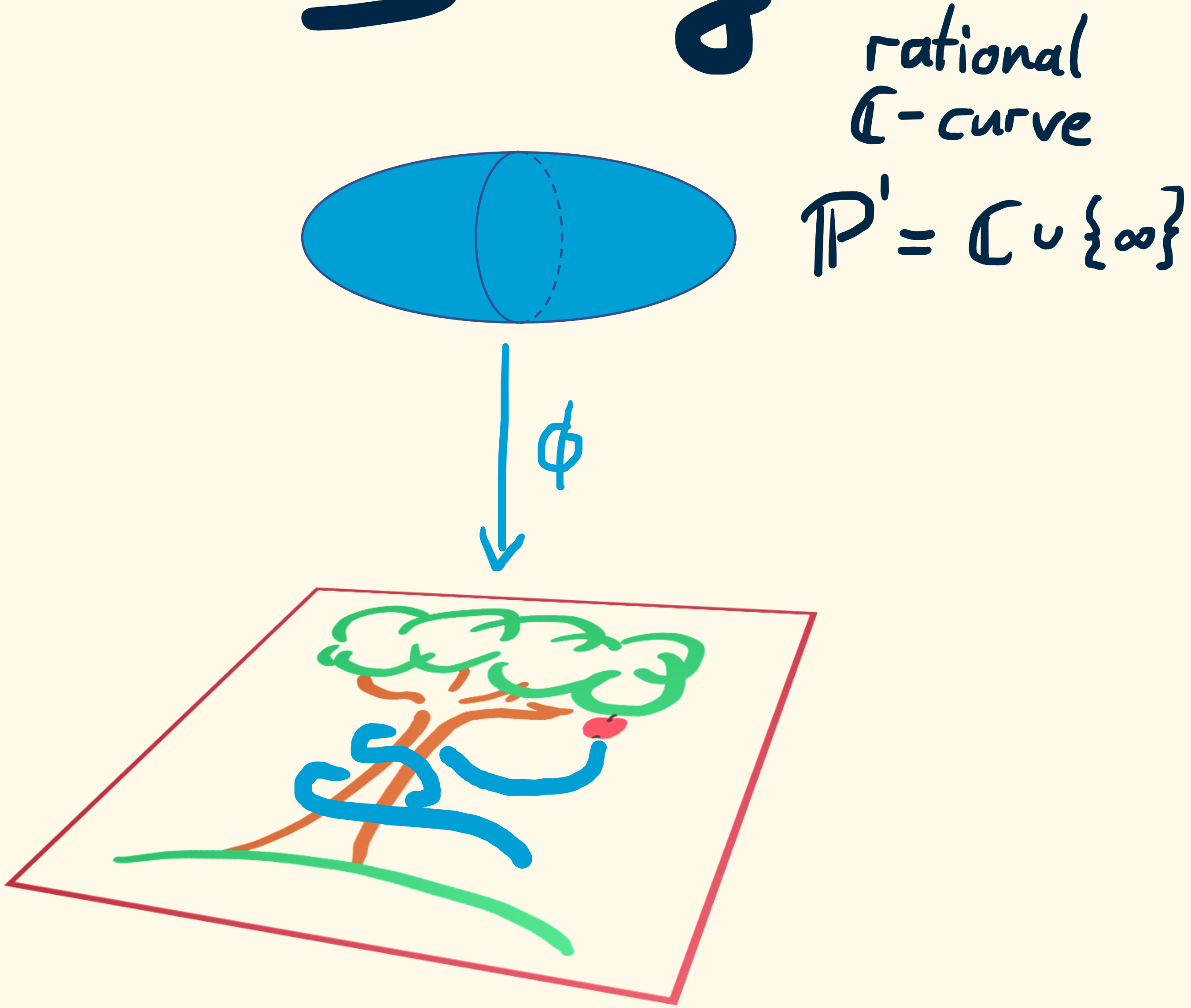

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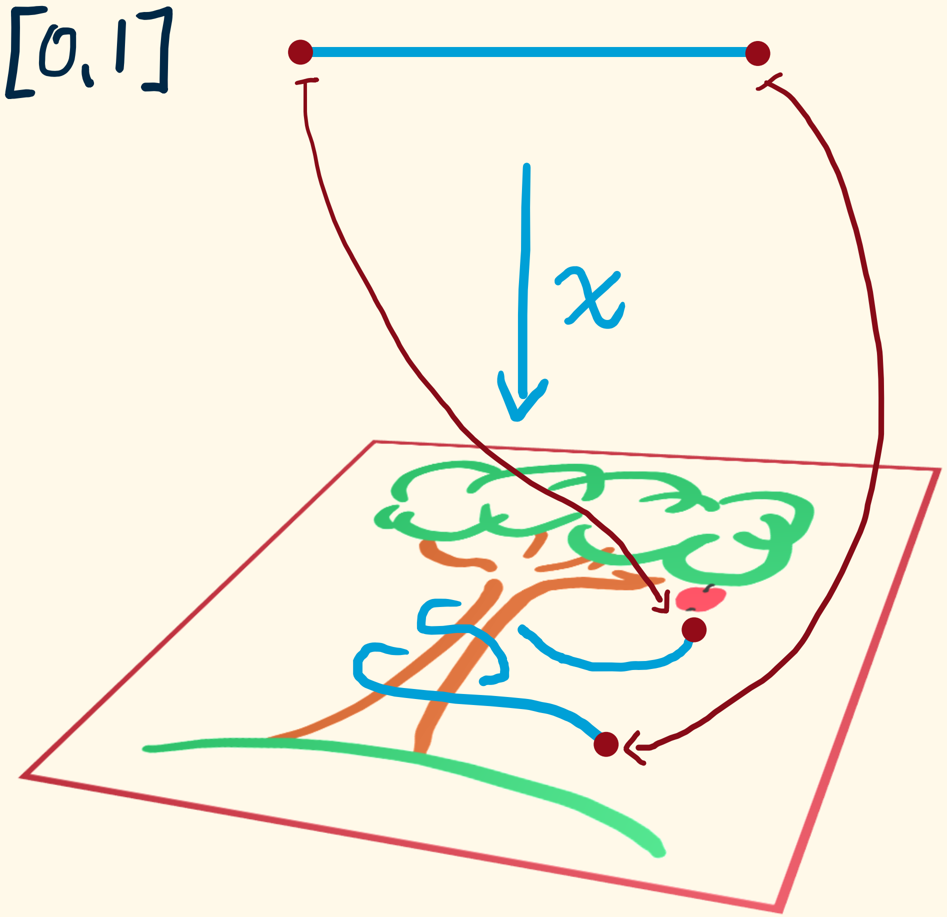

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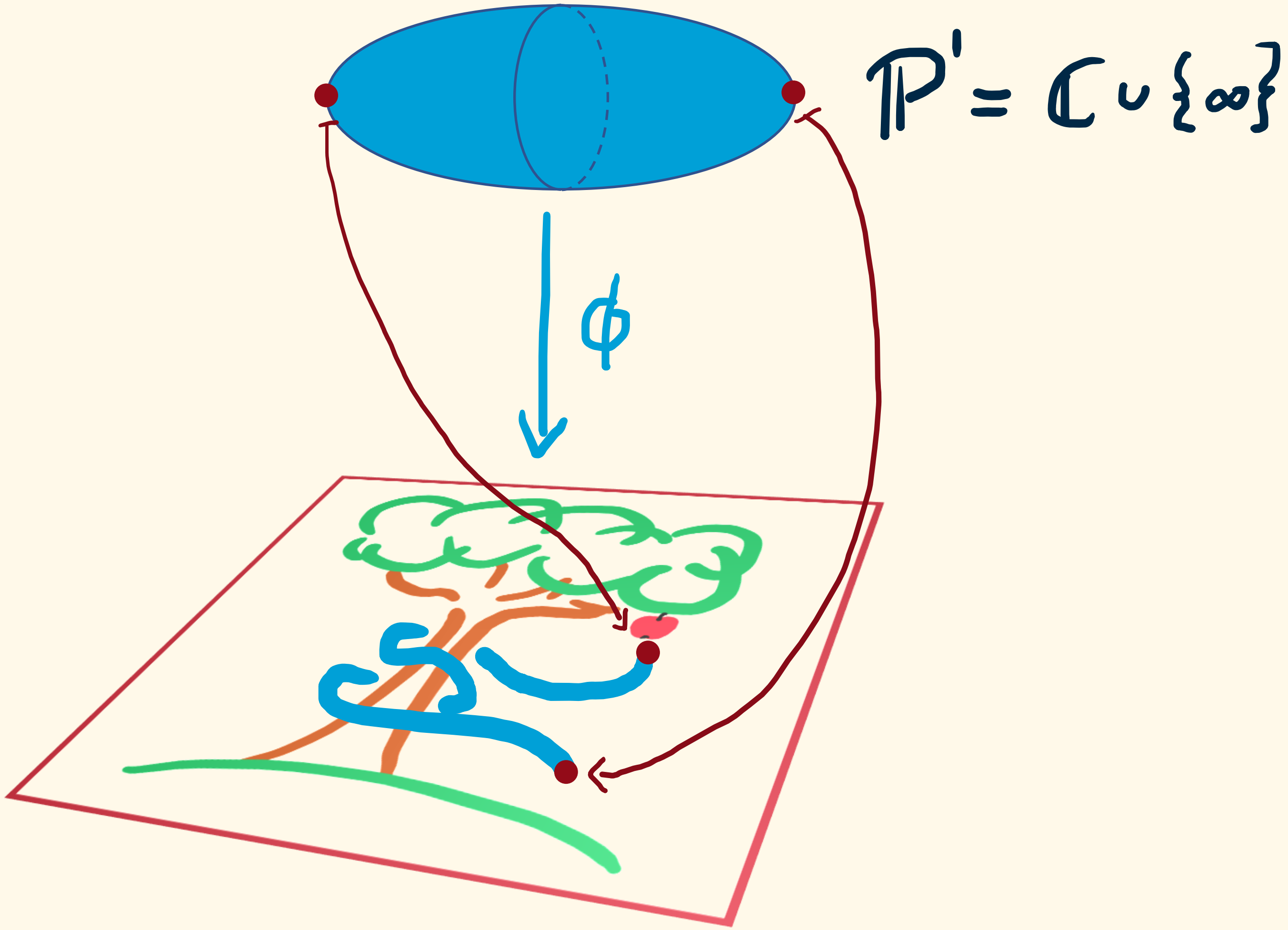
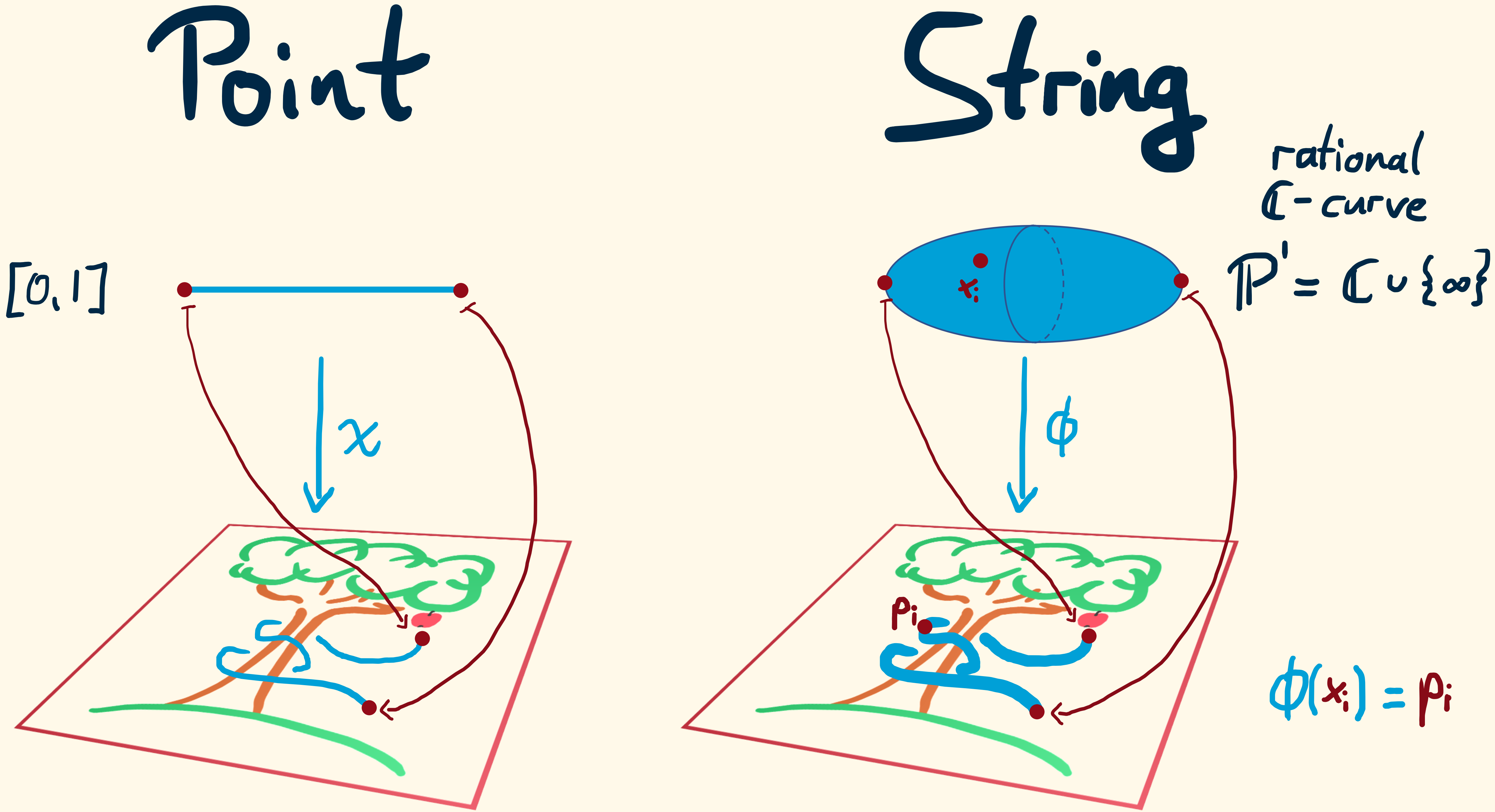
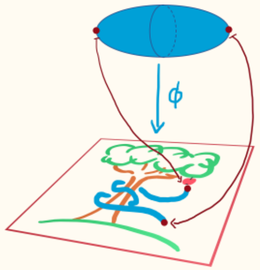



Point
String
rational
ℂ-curve
[0,1]
x
$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$
x_i
ϕ
p_i
$\phi(x_i) = p_i$

$\int \mathcal{D}\Phi$

{  }

$$\int \mathcal{D}\Phi$$

$\Big\{ \phi :$ Rat. Curve $\longrightarrow$ 

such that $\phi(x_i) = p_i \Big\}$

$$\int D\Phi \quad P_*(\Phi, Q)$$

$\{\phi:$ Rat. Curve $\longrightarrow$ 

such that $\phi(x_i) = p_i\}$

$$\int_{\left\{\substack{\phi : \text{Rat. Curve} \to \boxed{\quad} \\ \text{such that } \phi(x_i) = p_i}\right\}} \mathcal{D}\Phi \quad P_*(\Phi, Q)$$

$*$ = A-twisted $\mathcal{N}=(2,2)$ σ-model coupled to 2-dimensional topological gravity

$$\int_{\left\{\begin{array}{l}\phi:\ \text{Rat. Curve} \to \square \\ \text{such that } \phi(x_i)=p_i\end{array}\right\}} D\Phi \quad P_*(\Phi, Q)$$

$$\int_{\{\phi : \text{Rat. Curve} \to \square \text{ such that } \phi(x_i) = p_i\}} D\Phi \; P_*(\Phi, Q)$$

ϕ: Rat. Curve $\longrightarrow$ 

such that $\phi(x_i) = p_i$

+ SUSY

$$\int \mathcal{D}\Phi \quad P_*(\Phi, Q)$$

integrated over

$\{ \phi$: Rat. Curve $\longrightarrow$ 

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+ SUSY

Supersymmetric Localisation

$$= \sum Q^d \cdot \int \mathcal{D}\phi \quad 1$$

sum over degree d in 

integrated over

$\{ \phi$: Rat. Curve $\longrightarrow$ 

such that $\phi(x_i) = p_i$

ϕ is algebraic $\}$

$\int \mathcal{D}\Phi \quad P_*(\Phi, Q)$

$\{\phi$: Rat. Curve $\longrightarrow$ 
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+ SUSY

Well defined in Algebraic Geometry

Supersymmetric Localisation

$= \sum Q^d \cdot \int \mathcal{D}\phi \quad 1$

degree d in 

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$$\int_{\left\{\substack{\Phi:\ \text{Rat. Curve} \to \mathbb{P}^2 \\ \text{such that } \phi(x_i) = p_i}\right\} + \text{SUSY}} \mathcal{D}\Phi \quad P_*(\Phi, Q)$$

Well defined in Algebraic Geometry

Supersymmetric Localisation

$$= \sum_{\substack{\text{degree} \\ d \text{ in } \mathbb{P}^2}} Q^d \cdot \int_{\left\{\substack{\Phi:\ \text{Rat. Curve} \to \mathbb{P}^2 \\ \text{such that } \phi(x_i) = p_i \\ \phi \text{ is algebraic}}\right\}} \mathcal{D}\phi \quad 1$$

$$\int_{\left\{\begin{array}{l}\phi:\ \text{Rat. Curve} \to \mathbb{P}^2 \\ \text{such that } \phi(x_i) = p_i\end{array}\right\}} \mathcal{D}\Phi \quad P_*(\Phi, Q)$$

+ SUSY

Supersymmetric Localisation

$$= \sum_{d>0} Q^d \cdot N_d$$

Well defined in Algebraic Geometry

From Curves

↓

to Strings

From Curves

to Strings

...and back!

$$N_d = \#\left\{\begin{array}{l}\text{rational degree } d \ \mathcal{L} \\ \text{such that } p_1 \ldots p_{3d-1} \in \mathcal{L}\end{array}\right\}$$

d	1	2	3	4	5	>6
N_d	1	1	12	620	87304	Recursion
	antiquity		1853	1873	~1980	Kontsevich ~1994

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Kontsevich's recursion:

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_i>0}} N_{d_1} N_{d_2} d_1^2 d_2 \left(d_2 \binom{3d-4}{3d_1-2} - d_1 \binom{3d-4}{3d_1-1} \right)$$

Kontsevich's recursion:

$$N_d + \sum_{\substack{d_1+d_2=d \\ d_i>0}} N_{d_1} N_{d_2} d_1^3 d_2 \binom{3d-4}{3d_1-1} = \sum_{\substack{d_1+d_2=d \\ d_i>0}} N_{d_1} N_{d_2} d_1^2 d_2^2 \binom{3d-4}{3d_1-2}$$

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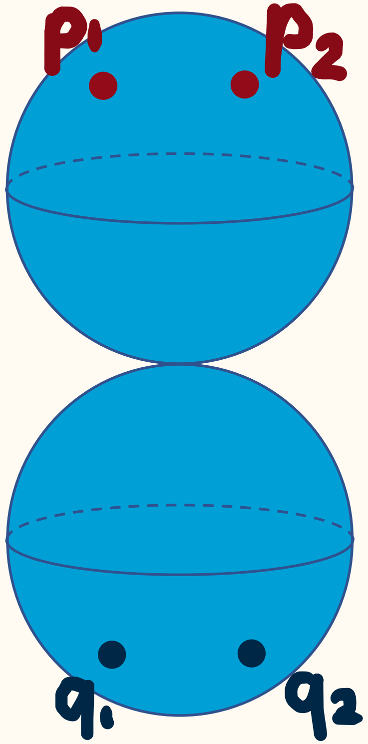

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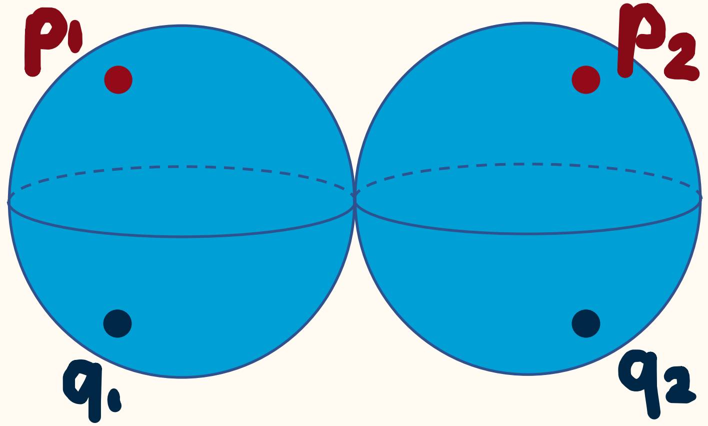

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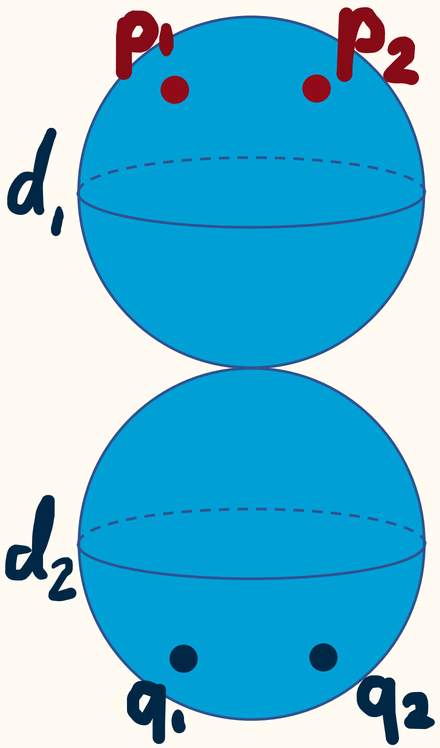

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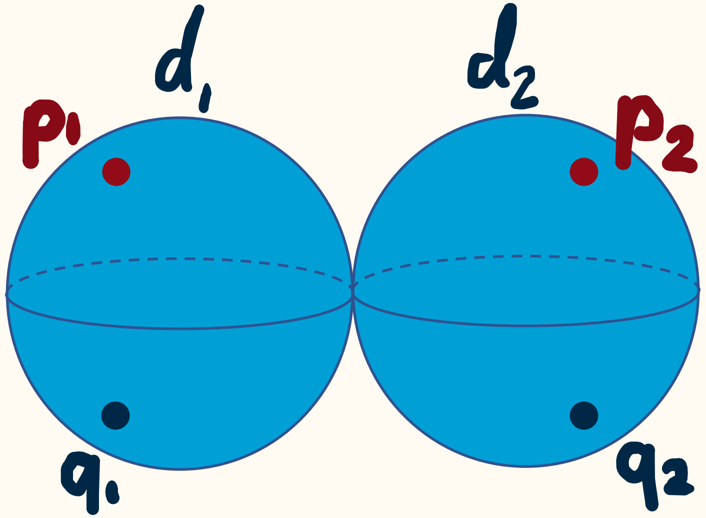

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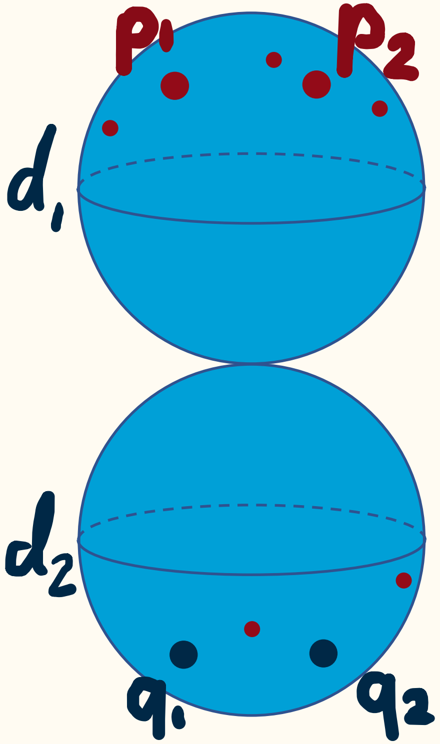

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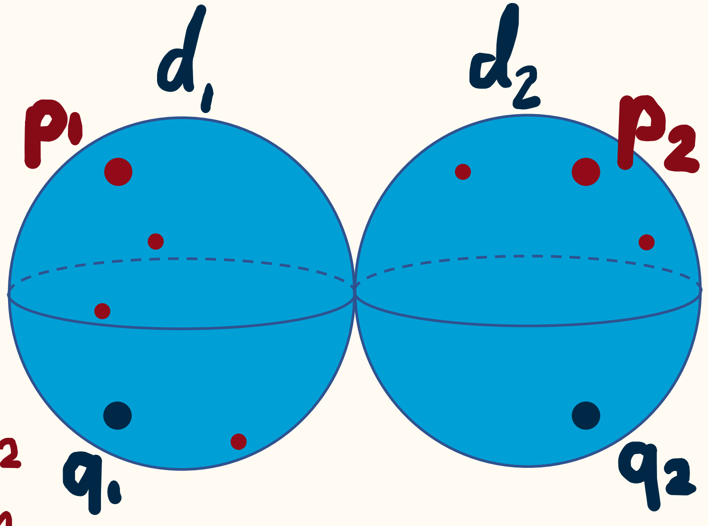

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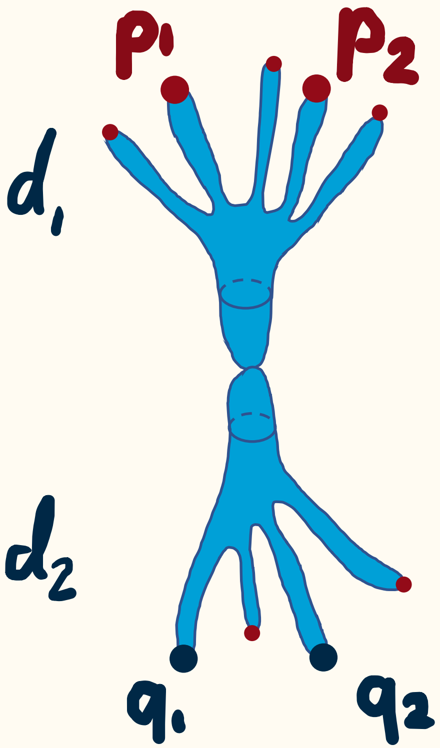

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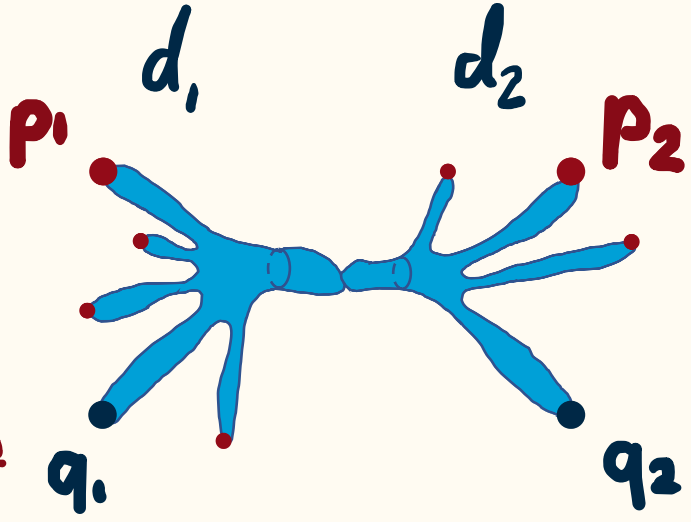

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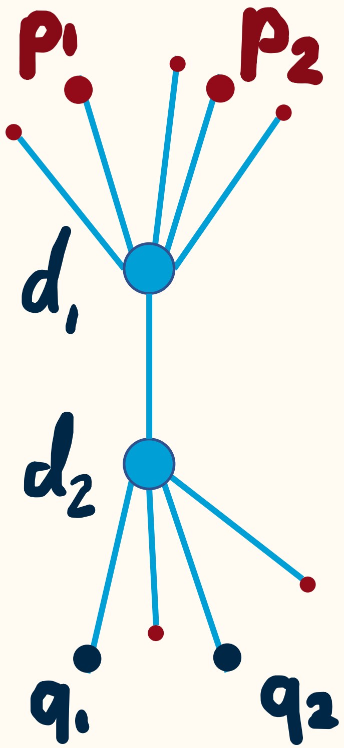

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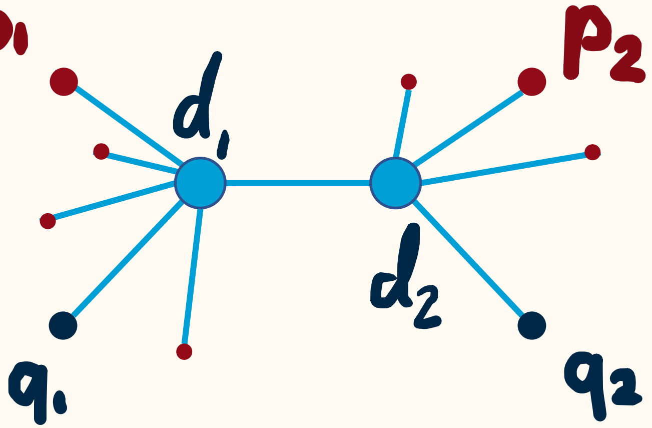

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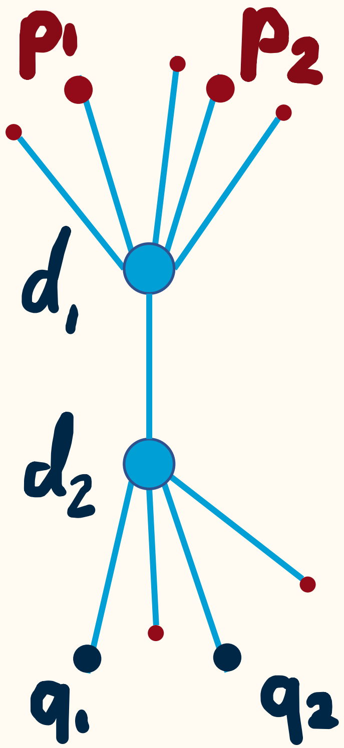

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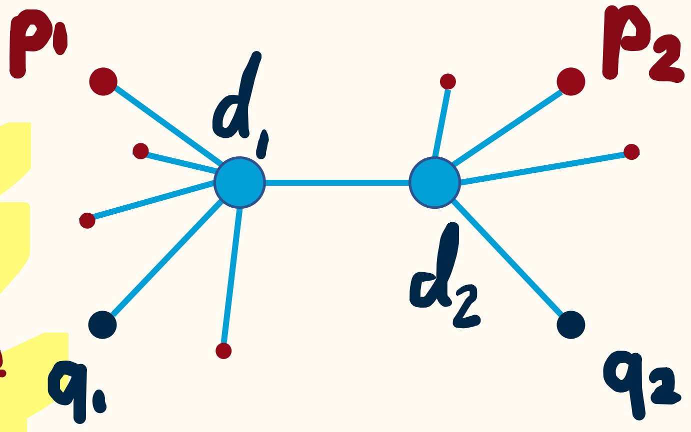

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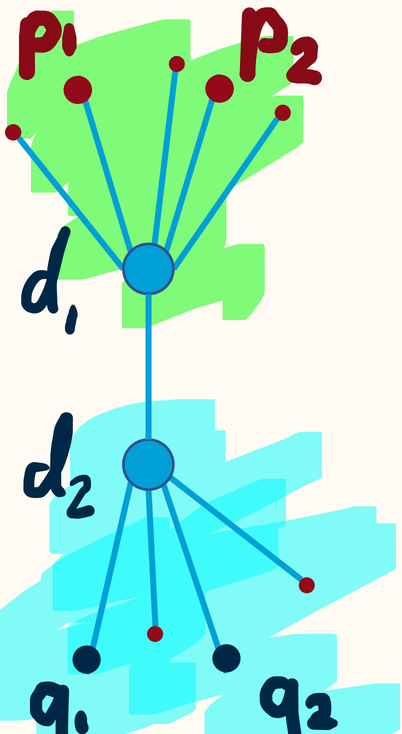

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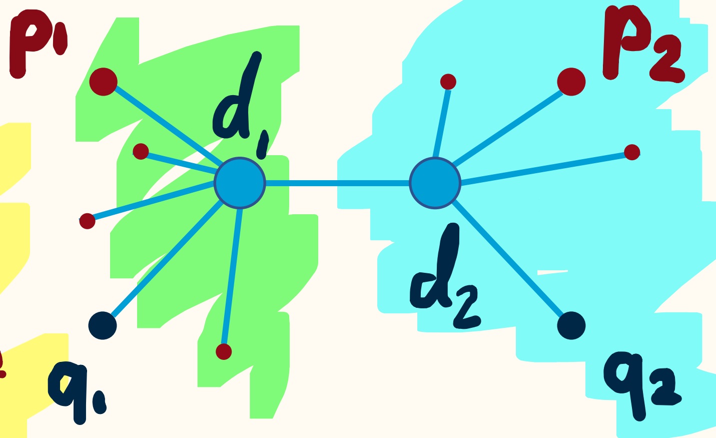
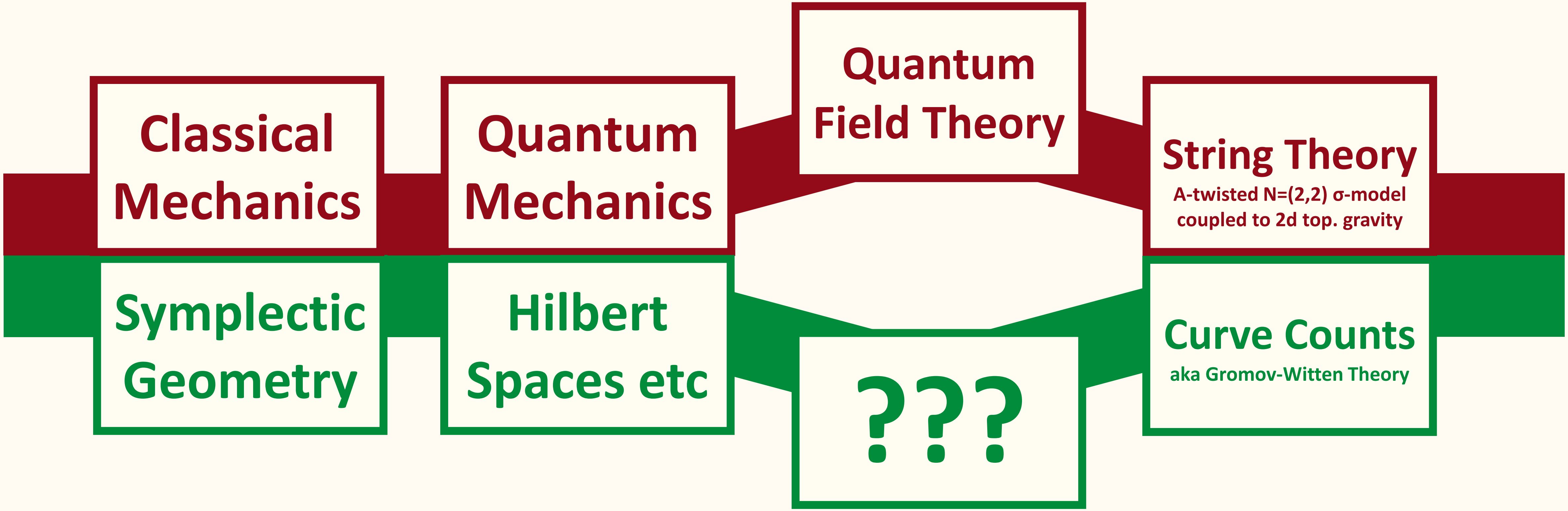


Physics



Mathematics

What we are now witnessing on the geometry/physics frontier is, in my opinion, one of the most refreshing events in the mathematics of the 20th century. The ramifications are vast and the ultimate nature and scope of what is being developed can barely be glimpsed. It might well come to dominate the mathematics of the 2lst century. [...] For those who are looking for a solid, safe PhD thesis, this field is hazardous, but for those who want excitement and action it must be irresistible.

Michael F. Atiyah. *Response to: "Theoretical mathematics: toward a cultural synthesis of mathematics and theoretical physics"*, by A. Jaffe and F. Quinn. Bull. Amer. Math. Soc. (N.S.) 30 (1994), 178-179.